Malaysian Journal of Mathematical Sciences 19(2): 443–469 (2025)
https://doi.org/10.47836/mjms.19.2.05

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

System of Caputo Fractional Differential Equations for Tuberculosis Disease with Effects of Immune and Asymptomatic Patients Classes: Theoretical and Analytical Solutions

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Received: 15 July 2024
Accepted: 9 November 2024

Abstract

Tuberculosis is an extremely serious disease that affects a large number of people around the world. In this paper, we investigate epidemic model of TB in the sense of Caputo derivative, integrating the impact of immune and asymptomatic classes. The primary challenge is accurately estimating the disease spread and assessing the effectiveness of human immunity. We establish the qualitative analysis of the constructed model, which include positivity, existence, and uniqueness of the solution, and Hyers–Ulam stability of the solution. The computed basic reproduction number $\mathbf{R}_0$ is used to obtain the normalized forward sensitivity index for each parameter with the purpose of identifying key parameters essential for the disease control. Numerical analysis are carried out utilizing the homotopy perturbation method and fractional differential transform method. The comparative study of integer order $\varrho = 1$ is done to validate the numerical performance. Our findings suggest that immune class and asymptomatic class individuals play a significant role in reducing/spreading TB prevalence and burden within human populations. We noticed a significant decline in the susceptible and asymptomatic classes as a result of self–immune and adequate treatment. These findings contribute to the management and control of the tuberculosis disease.

Keywords: Caputo fractional tuberculosis model; homotopy perturbation method; fractional differential transform method.

1 Introduction

Tuberculosis (TB) is a highly contagious bacterial infection caused by Mycobacterium tuberculosis. It poses a serious threat to public health and economic growth, especially in developing countries. TB is among the top 13 killer diseases worldwide, after the human immunodeficiency virus. It primarily affects the lungs but also can have an effect on other organs including the kidneys, bones, and main nervous system [1]. The study of infectious diseases has gained tremendous attention after Sir Ronald Ross research on malaria [9]. Waaler et al. [40] developed the first mathematical model of TB dynamics in 1960. The model focused on prediction and control strategies utilizing simulation methodologies. The authors proposed the systems of nonlinear ordinary differential equations to describe the dynamics of TB. They used a probabilistic technique to explain the linear relationship between infection rate and prevalence [34, 35]. Mathematical modelling of infectious diseases has become an essential tool for public health decision–makers, due to its prediction and ability to control the progression of diseases effectively [21].

In the recent time, there has been a shift towards using fractional order differential equations for more accurate modelling of real–world phenomena, improving predictive precision and capturing memory effects in dynamical systems including mrsa bacterial disease [13], HIV/AIDs [25], TB [37], and COVID–19 [33]. In the context of TB, memory effects are critical due to the long incubation period and latency before active disease state [11]. The epidemiological relevance of fractional order Caputo–type mathematical models provide a more detailed understanding of TB transmission dynamics than traditional compartmental models. By incorporating fractional derivatives, these models capture long–term memory effects, non–integer behaviors, and spatial heterogeneity, offering a realistic view of TB spread influenced by population mobility and social networks.

These models also improve predictive accuracy, aiding in forecasting TB epidemics and assessing the effectiveness of interventions like vaccination and treatment. Ultimately, they support evidence–based public health strategies for TB control and prevention [36, 28]. Panchal et al. [31] developed Caputo type SEITR TB model to analyze the disease dynamics. K.M. Owolabi and E. Pindza [30] analyzed the Caputo fractional order TB model to investigate the control measures. Avazzadeh et al. [5] investiaged the Caputo type fractional order TB model by using generalized Laguerre polynomials. Using fractional order model, the studies [14, 6] demonstrated its effectiveness by proving the solution existence and uniqueness. Atangana et al. [4] studied a novel fractional order model and analyzed the stability and uniqueness of the solutions.

Stability is a key factor in qualitative theory of differential equations, and finding the exact solution is often a complex and challenging task. Therefore, many numerical algorithms were developed to solve the problem. In this sense, we evaluate the stability of the provided problem. Literature contains a variety of stability types, including lyapunov, asymptotic, and exponential etc [17, 26]. However, Ulam stability [39], the most significant kind of stability was initially identified in 1940. After Ulam concept, Hyers [19] introduced Hyers–Ulam stability theory. Further, few researchers [12, 41] analyzed the stability of infectious diseases models by using Hyers–Ulam theory. Farman et al. [15] studied stability and sensitivity analysis of mathematical model for TB infection with vaccinated group.

Recently, a novel analytical approach has emerged for solving fractional differential equations known as the fractional differential transform method (FDTM), which formulates fractional power series. Elsaid [10] analyzed the application of FDTM with the combination of Adomian polynomials. Ibis et al. [20] investigated the fractional differential algebraic equations by FDTM. Furthermore, He [18] developed the homotopy perturbation method (HPM). Olayiwola and Adedokun

[29] utilized HPM method to analyze a novel Caputo fractional TB model with the treatment effects.

The main motivation behind this study is to analyze the dynamics of Caputo type fractional order TB model by utilizing fractional calculus principles and simulate the numerical solutions of the model via HPM and FDTM. This paper is compiled as follows. In Section 2, we describe the proposed model formulation. Section 3 discuss some fundamental definitions, corollary, and theorems required for the model analysis. In Section 4, we prove the existence and uniqueness of the model solution. Section 5 examines the disease–free and endemic equilibrium points and basic reproduction number. In Section 6, we conducts Hyers–Ulam stability analysis. Section 7 covers sensitivity analysis, which assesses the impact of model parameters on disease dynamics. In Section 8, the HPM and FDTM are used to analyze the fractional model numerically. Additionally, we accomplish numerical simulations to evaluate the impact of the solutions across various fractional order values and model classes on the TB transmission dynamics. Finally, Section 9 conclude the entire research.

2 Model Description

In this research, a fractional order model to describe the transmission dynamics of TB is proposed. The model incorporates fractional derivatives to capture the memory effects and non-integer order behaviors inherent in TB epidemiology. This paper distinctive novelty includes fractional derivatives, multi–component structures, data integration, sensitivity analysis, and novel mathematical techniques. These innovations collectively contribute to the TB model's originality, making it a more comprehensive and precise tool for understanding and controlling the disease.

The total population at time $(\zeta, \mathbf{N}(\zeta))$ is dividing into seven compartments representing different stages including susceptible class $\mathbf{S}(\zeta)$, the immunized class $\mathbf{M}(\zeta)$, the exposed class $\mathbf{E}(\zeta)$, the asymptomatic class $\mathbf{A}(\zeta)$, the infected class $\mathbf{I}(\zeta)$, the treatment class $\mathbf{T}(\zeta)$, and the recovered class $\mathbf{R}(\zeta)$. The model provides insights into the spread and control of TB in the population. This fractional–order epidemic model is significant because it can better describe the complex dynamics of tuberculosis transmission than typical compartmental models. Moreover, the model can incorporate heterogeneity in TB transmission, accounting for factors such as population demographics, social interactions, and spatial distributions. The fractional order **SMEAITR** model is considered as follows:

$$\begin{cases} {}^{C}D_0^{\varrho}\mathbf{S}(\zeta) = (1-\pi^{\varrho})\Lambda^{\varrho} + \rho^{\varrho}\mathbf{M} + (1-r_1)\lambda^{\varrho}\mathbf{E} + \sigma^{\varrho}\mathbf{R} - \alpha^{\varrho}\mathbf{S}(\mathbf{I}+\beta^{\varrho}\mathbf{T}) - \eta^{\varrho}\mathbf{S}, \\ {}^{C}D_0^{\varrho}\mathbf{M}(\zeta) = \pi^{\varrho}\Lambda^{\varrho} - (\rho^{\varrho}+\eta^{\varrho})\mathbf{M}, \\ {}^{C}D_0^{\varrho}\mathbf{E}(\zeta) = \alpha^{\varrho}\mathbf{S}(\mathbf{I}+\beta^{\varrho}\mathbf{T}) - (\lambda^{\varrho}+\gamma^{\varrho}+\omega^{\varrho}+\eta^{\varrho})\mathbf{E}, \\ {}^{C}D_0^{\varrho}\mathbf{A}(\zeta) = r_1\lambda^{\varrho}\mathbf{E} - (\theta^{\varrho}+\eta^{\varrho})\mathbf{A}, \\ {}^{C}D_0^{\varrho}\mathbf{I}(\zeta) = \omega^{\varrho}\mathbf{E} + \theta^{\varrho}\mathbf{A} - (\xi^{\varrho}+\psi_1^{\varrho}+\eta^{\varrho})\mathbf{I}, \\ {}^{C}D_0^{\varrho}\mathbf{T}(\zeta) = \xi^{\varrho}\mathbf{I} - (\mu^{\varrho}+\psi_2^{\varrho}+\eta^{\varrho})\mathbf{T}, \\ {}^{C}D_0^{\varrho}\mathbf{R}(\zeta) = \gamma^{\varrho}\mathbf{E} + \mu^{\varrho}\mathbf{T} - (\sigma^{\varrho}+\eta^{\varrho})\mathbf{R}, \end{cases} \tag{1}$$

with $\mathbf{N} = \mathbf{S}+\mathbf{M}+\mathbf{E}+\mathbf{A}+\mathbf{I}+\mathbf{T}+\mathbf{R}$. The parameters $(1-\pi)\Lambda$ and $\pi\Lambda$ represent the rate of recruitment that impacts a susceptible and immunized class respectively. Additionally, ρ indicates the rate of drug discontinuation. Natural death rate is denoted by η. The infectious rate of individuals of susceptible class $\mathbf{S}$ from infected class $\mathbf{I}$ due to close mutual contact is represented by α, and β denotes the decrease of infectiousness. γ shows the contagious predisposition treat-

ment effectiveness rate. ω is the rate at which innate TB evolves too highly infectious. The rate of progression from $\mathbf{E}$ to $\mathbf{A}$ is symbolized by r_1, and λ denotes the latency or incubation period.

The rate of people migrating from $\mathbf{A}$ to $\mathbf{I}$ is represented by θ. The symbol ξ reflects successful treatment of infectious TB patients, whereas ψ_1 represents the disease-induced mortality rate in $\mathbf{I}$. The TB–induced death rate in $\mathbf{T}$ is given by ψ_2. The rate at which recovered TB patients progress from $\mathbf{T}$ to $\mathbf{R}$ is denoted by μ. Finally, σ represents the rate of recovered people becoming sensitive again and rejoining the susceptible class. The schematic diagram of the dynamical Model (1) is depicted in Figure 1.

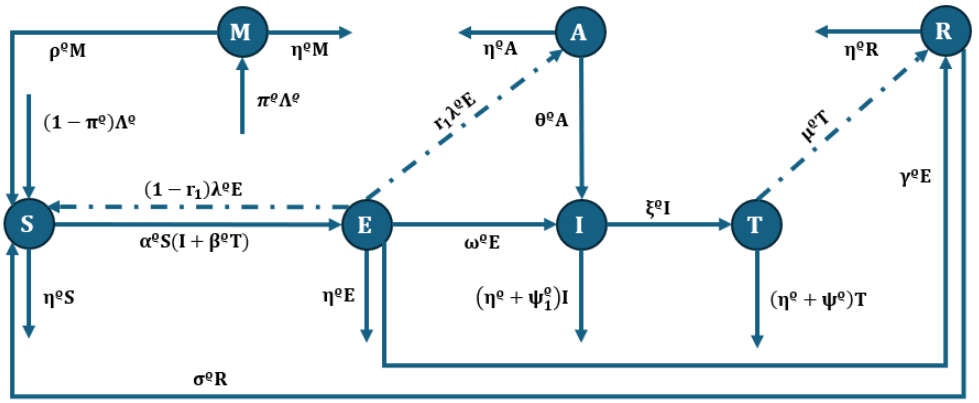


Figure 1: Diagram of Model (1).

The outward arrow symbolizes the terms exiting the compartment, whereas inward arrow signifies the term entering the compartment. It is reasonable to assume that for $\zeta \geq 0$, all variables are greater than or equal to zero.

For simplicity Model (1) can be written as,

$$\begin{cases} {}^{C}D_0^{\varrho}\mathbf{S}(\zeta) = (1-\pi^{\varrho})\Lambda^{\varrho} + \rho^{\varrho}\mathbf{M} + (1-r_1)\lambda^{\varrho}\mathbf{E} + \sigma^{\varrho}\mathbf{R} - \alpha^{\varrho}\mathbf{S}(\mathbf{I}+\beta^{\varrho}\mathbf{T}) - \eta^{\varrho}\mathbf{S}, \\ {}^{C}D_0^{\varrho}\mathbf{M}(\zeta) = \pi^{\varrho}\Lambda^{\varrho} - k_1\mathbf{M}, \\ {}^{C}D_0^{\varrho}\mathbf{E}(\zeta) = \alpha^{\varrho}\mathbf{S}(\mathbf{I}+\beta^{\varrho}\mathbf{T}) - k_2\mathbf{E}, \\ {}^{C}D_0^{\varrho}\mathbf{A}(\zeta) = r_1\lambda^{\varrho}\mathbf{E} - k_3\mathbf{A}, \\ {}^{C}D_0^{\varrho}\mathbf{I}(\zeta) = \omega^{\varrho}\mathbf{E} + \theta^{\varrho}\mathbf{A} - k_4\mathbf{I}, \\ {}^{C}D_0^{\varrho}\mathbf{T}(\zeta) = \xi^{\varrho}\mathbf{I} - k_5\mathbf{T}, \\ {}^{C}D_0^{\varrho}\mathbf{R}(\zeta) = \gamma^{\varrho}\mathbf{E} + \mu^{\varrho}\mathbf{T} - k_6\mathbf{R}, \end{cases} \tag{2}$$

where

$$k_1 = \rho^{\varrho} + \eta^{\varrho}, \qquad k_2 = \lambda^{\varrho} + \gamma^{\varrho} + \omega^{\varrho} + \eta^{\varrho}, \qquad k_3 = \theta^{\varrho} + \eta^{\varrho},$$
$$k_4 = \xi^{\varrho} + \psi_1^{\varrho} + \eta^{\varrho}, \qquad k_5 = \mu^{\varrho} + \psi_2^{\varrho} + \eta^{\varrho}, \qquad k_6 = \sigma^{\varrho} + \eta^{\varrho}.$$

In the above model, the fractional order derivatives denoted by ${}^{C}D^{\varrho}$ where $\varrho \in (0,1]$ are the Caputo derivatives associated with biological parameters.

3 Preliminaries

Definition 3.1. [22] *Let $\Gamma(.)$ is the gamma function and the fractional integral of order $\varrho > 0$ of a continuous function $g : R^+ \to R$ is defined as:*

$$I^{\varrho} g(\zeta) = \frac{1}{\Gamma(\varrho)} \int_0^{\zeta} (\zeta - \vartheta)^{\varrho - 1} g(\vartheta) d\vartheta.$$

Definition 3.2. [8] *The relative change in the variable divided by the relative change in the parameter is the normalized forward sensitivity index of a quantity of interest $\mathbf{R}_0$ to a parameter of interest γ, is defined by,*

$$\Psi_{\gamma}^{\mathbf{R}_0} = \frac{\partial \mathbf{R}_0}{\partial \gamma} \frac{\gamma}{\mathbf{R}_0}.$$

Definition 3.3. [2] *The Caputo derivative Laplace transformation is defined as:*

$$\mathfrak{L}[{}^{C}D^{\varrho} g(\zeta)] = \vartheta^{\varrho} G(\vartheta) - \sum_{\tau=0}^{m-1} \vartheta^{\varrho - \tau - 1} g^{\tau}(0), \quad m - 1 < \varrho < m, \; m \in N.$$

Definition 3.4. [3] *The Caputo fractional order ϱ derivative of a function $f \in C^{(\rho)}((0, \infty), R)$ is defined as:*

$${}^{C}D_0^{\varrho} f(\zeta) = \frac{1}{\Gamma(\rho - \varrho)} \int_{\zeta_0}^{\zeta} (\zeta - \vartheta)^{\rho - \varrho - 1} f^{(\rho)}(\vartheta) d\vartheta, \tag{3}$$

where $\rho = [\varrho] + 1$ and $[\varrho]$ is integer part of ϱ.

We expand the continuous function $f(\zeta)$ in the form of a fractional power series is,

$$f(\zeta) = \sum_{\tau=0}^{\infty} F(\tau)(\zeta - \zeta_0)^{\frac{\tau}{\varrho}}, \tag{4}$$

the fractional differential transform of $f(\zeta)$ is denoted by $F(\tau)$, where ϱ is the order of fraction. To avoid fractional initial and boundary conditions, we define the fractional derivative via Caputo definition. The relationship between the Riemann–Liouville and Caputo operators is given by,

$$D_{*\zeta_0}^{\varrho} f(\zeta) = D^{\varrho}{}_{\zeta_0} \cdot \left[f(\zeta) - \sum_{\tau=0}^{\rho-1} \frac{1}{\tau!} (\zeta - \zeta_0)^{\tau} f^{(\tau)}(\zeta_0) \right]. \tag{5}$$

Setting $f(\zeta) = f(\zeta) - \sum_{\tau=0}^{\rho-1} \frac{1}{\tau!} (\zeta - \zeta_0)^{\tau} f^{(\tau)}(\zeta_0)$ in (4) and using (5), the fractional derivative in the Caputo sense [7] is,

$$D_{*\zeta_0}^{\varrho} f(\zeta) = \frac{1}{\Gamma(\rho - \varrho)} \frac{d^{\rho}}{d\zeta^{\rho}} \int_{\zeta_0}^{\zeta} \frac{f(\vartheta) - \sum_{\tau=0}^{\rho-1} \frac{1}{\tau!} (\vartheta - \zeta_0)^{\tau} f^{(\tau)}(\zeta_0)}{(\zeta - \vartheta)^{1 + \varrho - \rho}} \, d\vartheta.$$

As the initial conditions are applied to the derivatives of integer orders, the transformation of the initial conditions can be expressed as follows:

$$F(\tau) = \begin{cases} \dfrac{1}{\left(\frac{\tau}{m}\right)!} \dfrac{d^{\frac{\tau}{m}} f(\zeta)}{d\zeta^{\frac{\tau}{m}}} \Bigg|_{\zeta = \zeta_0}, & \text{if } \dfrac{\tau}{m} \in Z^+, \text{ for } \tau = 0, 1, 2, \ldots, (m\varrho - 1), \\ 0, & \text{if } \dfrac{\tau}{m} \notin Z^+, \end{cases}$$

where ϱ is fractional order, and m is to be chosen as a positive integer.

Let $H(\tau)$ and $G(\tau)$ be the Laplace transformation of $h(\zeta)$ and $g(\zeta)$, respectively then we have the following existing theorems.

Theorem 3.1. [20] *If* $h(\zeta) = f(\zeta) \pm g(\zeta)$ *then* $H(\tau) = F(\tau) \pm G(\tau)$.

Theorem 3.2. [20] *If* $h(\zeta) = f(\zeta) \cdot g(\zeta)$ *then* $H(\tau) = \sum_{l=0}^{\tau} F(l) \cdot G(\tau - l)$.

Theorem 3.3. [20] *If* $h(\zeta) = (\zeta - \zeta_0)^{\varrho}$ *then* $H(\tau) = \delta(\tau - m\varrho)$ *where,*

$$\delta(\tau) = \begin{cases} 1, & \tau = 0, \\ 0, & \tau \neq 0. \end{cases} \tag{6}$$

Theorem 3.4. [20] *If* $f(\zeta) = D^{\varrho}_{\zeta_0}[h(\zeta)]$ *then* $F(\tau) = \dfrac{\Gamma\left(\varrho+1+\frac{\tau}{m}\right)}{\Gamma\left(1+\frac{\tau}{m}\right)} H(\tau + m\varrho)$.

Corollary 3.1. [16] *Let* $g(\zeta) \in C[c,d]$, ${}^{C}D^{\varrho}g(\zeta) \in C[c,d]$, *and* $0 < \varrho \leq 1$. *If,*

$$\begin{cases} {}^{C}D^{\varrho}g(\zeta) \geq 0, & \text{then } g(\zeta) \text{ is non–decreasing}, \\ {}^{C}D^{\varrho}g(\zeta) \leq 0, & \text{then } g(\zeta) \text{ is non–increasing}, \end{cases}$$

for all $\zeta \in (c,d)$.

4 Theoretical Analysis

4.1 Existence and uniqueness

This part demonstrate the existence and uniqueness of the system (2). First, we have,

$$\begin{cases} {}^{C}D^{\varrho}_{0}\mathbf{S}(\zeta) = \mathcal{G}_1 = (1-\pi^{\varrho})\Lambda^{\varrho} + \rho^{\varrho}\mathbf{M} + (1-r_1)\lambda^{\varrho}\mathbf{E} + \sigma^{\varrho}\mathbf{R} - \alpha^{\varrho}\mathbf{S}(\mathbf{I} + \beta^{\varrho}\mathbf{T}) - \eta^{\varrho}\mathbf{S}, \\ {}^{C}D^{\varrho}_{0}\mathbf{M}(\zeta) = \mathcal{G}_2 = \pi^{\varrho}\Lambda^{\varrho} - k_1\mathbf{M}, \\ {}^{C}D^{\varrho}_{0}\mathbf{E}(\zeta) = \mathcal{G}_3 = \alpha^{\varrho}\mathbf{S}(\mathbf{I} + \beta^{\varrho}\mathbf{T}) - k_2\mathbf{E}, \\ {}^{C}D^{\varrho}_{0}\mathbf{A}(\zeta) = \mathcal{G}_4 = r_1\lambda^{\varrho}\mathbf{E} - k_3\mathbf{A}, \\ {}^{C}D^{\varrho}_{0}\mathbf{I}(\zeta) = \mathcal{G}_5 = \omega^{\varrho}\mathbf{E} + \theta^{\varrho}\mathbf{A} - k_4\mathbf{I}, \\ {}^{C}D^{\varrho}_{0}\mathbf{T}(\zeta) = \mathcal{G}_6 = \xi^{\varrho}\mathbf{I} - k_5\mathbf{T}, \\ {}^{C}D^{\varrho}_{0}\mathbf{R}(\zeta) = \mathcal{G}_7 = \gamma^{\varrho}\mathbf{E} + \mu^{\varrho}\mathbf{T} - k_6\mathbf{R}, \end{cases} \tag{7}$$

with initial condition $\mathbf{S}(0) = \mathbf{S}_0$, $\mathbf{M}(0) = \mathbf{M}_0$, $\mathbf{E}(0) = \mathbf{E}_0$, $\mathbf{A}(0) = \mathbf{A}_0$, $\mathbf{I}(0) = \mathbf{I}_0$, $\mathbf{T}(0) = \mathbf{T}_0$, $\mathbf{R}(0) = \mathbf{R}_0$ and $\mathcal{G}_i \;\; i = 1, \ldots, 7$ are singular kernels. From (7), we have,

$$\begin{cases} {}^{C}D^{\varrho}\mathcal{H}(\zeta) = \Psi(\zeta, \mathcal{H}(\zeta)), \\ \qquad \mathcal{H}(0) = \mathcal{H}_0 \geq 0, \;\; 0 \leq \zeta \leq T < \infty, \;\; 0 < \varrho \leq 1, \end{cases} \tag{8}$$

where $\mathcal{H}(\zeta) = (\mathbf{S}, \mathbf{M}, \mathbf{E}, \mathbf{A}, \mathbf{I}, \mathbf{T}, \mathbf{R})^T$, $\mathcal{H}(0) = (\mathbf{S}_0, \mathbf{M}_0, \mathbf{E}_0, \mathbf{A}_0, \mathbf{I}_0, \mathbf{T}_0, \mathbf{R}_0)^T$. Using the integral form of the both sides of (8), we obtain,

$$\mathcal{H}(\zeta) - \mathcal{H}(0) = \frac{1}{\Gamma(\varrho)} \int_0^{\zeta} \Psi(\vartheta, \mathcal{H}(\vartheta))(\zeta - \vartheta)^{\varrho - 1} d\vartheta. \tag{9}$$

Let us write (9) for each class of the proposed model as,

$$\begin{cases} \mathbf{S}(\zeta)-\mathbf{S}(0)=\dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta(\zeta-\vartheta)^{\varrho-1}[\mathcal{G}_1(\vartheta,\mathbf{S}(\vartheta))]d\vartheta,\\ \mathbf{M}(\zeta)-\mathbf{M}(0)=\dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta(\zeta-\vartheta)^{\varrho-1}[\mathcal{G}_2(\vartheta,\mathbf{M}(\vartheta))]d\vartheta,\\ \mathbf{E}(\zeta)-\mathbf{E}(0)=\dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta(\zeta-\vartheta)^{\varrho-1}[\mathcal{G}_3(\vartheta,\mathbf{E}(\vartheta))]d\vartheta,\\ \mathbf{A}(\zeta)-\mathbf{A}(0)=\dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta(\zeta-\vartheta)^{\varrho-1}[\mathcal{G}_4(\vartheta,\mathbf{A}(\vartheta))]d\vartheta,\\ \mathbf{I}(\zeta)-\mathbf{I}(0)=\dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta(\zeta-\vartheta)^{\varrho-1}[\mathcal{G}_5(\vartheta,\mathbf{I}(\vartheta))]d\vartheta,\\ \mathbf{T}(\zeta)-\mathbf{T}(0)=\dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta(\zeta-\vartheta)^{\varrho-1}[\mathcal{G}_6(\vartheta,\mathbf{T}(\vartheta))]d\vartheta,\\ \mathbf{R}(\zeta)-\mathbf{R}(0)=\dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta(\zeta-\vartheta)^{\varrho-1}[\mathcal{G}_7(\vartheta,\mathbf{R}(\vartheta))]d\vartheta. \end{cases} \tag{10}$$

Theorem 4.1. *All the kernels $\mathcal{G}_i$ in the system (7), there are $\mathcal{H}_i>0,\ \ i=1,\dots,7$, therefore,*

$$\begin{cases} \|\mathcal{G}_1(\zeta,\mathbf{S}(\zeta))-\mathcal{G}_1(\zeta,\mathbf{S}_1(\zeta))\|\le\mathcal{H}_1\|\mathbf{S}(\zeta)-\mathbf{S}_1(\zeta)\|,\\ \|\mathcal{G}_2(\zeta,\mathbf{M}(\zeta))-\mathcal{G}_2(\zeta,\mathbf{M}_1(\zeta))\|\le\mathcal{H}_2\|\mathbf{M}(\zeta)-\mathbf{M}_1(\zeta)\|,\\ \|\mathcal{G}_3(\zeta,\mathbf{E}(\zeta))-\mathcal{G}_3(\zeta,\mathbf{E}_1(\zeta))\|\le\mathcal{H}_3\|\mathbf{E}(\zeta)-\mathbf{E}_1(\zeta)\|,\\ \|\mathcal{G}_4(\zeta,\mathbf{A}(\zeta))-\mathcal{G}_4(\zeta,\mathbf{A}_1(\zeta))\|\le\mathcal{H}_4\|\mathbf{A}(\zeta)-\mathbf{A}_1(\zeta)\|,\\ \|\mathcal{G}_5(\zeta,\mathbf{I}(\zeta))-\mathcal{G}_5(\zeta,\mathbf{I}_1(\zeta))\|\le\mathcal{H}_5\|\mathbf{I}(\zeta)-\mathbf{I}_1(\zeta)\|,\\ \|\mathcal{G}_6(\zeta,\mathbf{T}(\zeta))-\mathcal{G}_6(\zeta,\mathbf{T}_1(\zeta))\|\le\mathcal{H}_6\|\mathbf{T}(\zeta)-\mathbf{T}_1(\zeta)\|,\\ \|\mathcal{G}_7(\zeta,\mathbf{R}(\zeta))-\mathcal{G}_7(\zeta,\mathbf{R}_1(\zeta))\|\le\mathcal{H}_7\|\mathbf{R}(\zeta)-\mathbf{R}_1(\zeta)\|, \end{cases}$$

and contraction for $0\le\mathcal{H}_i<1,\ \ i=1,\dots,7$.

Proof. $\mathbf{S}$ and $\mathbf{S}_1$ be two continuous functions are considered, then by triangular inequality, we have,

$$\begin{aligned} \|\mathcal{G}_1(\zeta,\mathbf{S}(\zeta))-\mathcal{G}_1(\zeta,\mathbf{S}_1(\zeta))\| &= \|\alpha^\varrho(\mathbf{I}+\beta^\varrho\mathbf{T})(\mathbf{S}(\zeta)-\mathbf{S}_1(\zeta))-\eta^\varrho(\mathbf{S}(\zeta)-\mathbf{S}_1(\zeta))\|\\ &\le\|(\alpha^\varrho(d_1+\beta^\varrho d_2)-\eta^\varrho)(\mathbf{S}(\zeta)-\mathbf{S}_1(\zeta)\|\\ &\le|\alpha^\varrho(d_1+\beta^\varrho d_2)-\eta^\varrho|\|\mathbf{S}(\zeta)-\mathbf{S}_1(\zeta)\|\\ &\le\mathcal{H}_1\|\mathbf{S}(\zeta)-\mathbf{S}_1(\zeta)\|, \end{aligned} \tag{11}$$

where $\mathcal{H}_1=|\alpha^\varrho(d_1+\beta^\varrho d_2)-\eta^\varrho|,\ \|\mathbf{I}(\zeta)\|\le d_1,\ \|\mathbf{T}(\zeta)\|\le d_2$. Therefore $\mathcal{G}_1$ satisfies the Lipschitz condition. And if $0\le\mathcal{H}_1<1$, then it is also a contraction.

In the similar way, we have

$$\begin{cases} \|\mathcal{G}_2(\zeta,\mathbf{M}(\zeta))-\mathcal{G}_2(\zeta,\mathbf{M}_1(\zeta))\|\le\mathcal{H}_2\|\mathbf{M}(\zeta)-\mathbf{M}_1(\zeta)\|,\\ \|\mathcal{G}_3(\zeta,\mathbf{E}(\zeta))-\mathcal{G}_3(\zeta,\mathbf{E}_1(\zeta))\|\le\mathcal{H}_3\|\mathbf{E}(\zeta)-\mathbf{E}_1(\zeta)\|,\\ \|\mathcal{G}_4(\zeta,\mathbf{A}(\zeta))-\mathcal{G}_4(\zeta,\mathbf{A}_1(\zeta))\|\le\mathcal{H}_4\|\mathbf{A}(\zeta)-\mathbf{A}_1(\zeta)\|,\\ \|\mathcal{G}_5(\zeta,\mathbf{I}(\zeta))-\mathcal{G}_5(\zeta,\mathbf{I}_1(\zeta))\|\le\mathcal{H}_5\|\mathbf{I}(\zeta)-\mathbf{I}_1(\zeta)\|,\\ \|\mathcal{G}_6(\zeta,\mathbf{T}(\zeta))-\mathcal{G}_6(\zeta,\mathbf{T}_1(\zeta))\|\le\mathcal{H}_6\|\mathbf{T}(\zeta)-\mathbf{T}_1(\zeta)\|,\\ \|\mathcal{G}_7(\zeta,\mathbf{R}(\zeta))-\mathcal{G}_7(\zeta,\mathbf{R}_1(\zeta))\|\le\mathcal{H}_7\|\mathbf{R}(\zeta)-\mathbf{R}_1(\zeta)\|, \end{cases}$$

where $\mathcal{H}_2 = k_1,\ \mathcal{H}_3 = k_2,\ \mathcal{H}_4 = k_3,\ \mathcal{H}_5 = k_4,\ \mathcal{H}_6 = k_5,\ \mathcal{H}_7 = k_6$.

Hence all $\mathcal{G}_i$ satisfy the Lipschitz condition, and they are also contraction if $\mathcal{H}_i \in [0,1)$, $i = 2,\ldots,7$. From (10), the recursive pattern for $\mathbf{S},\ \mathbf{M},\ \mathbf{E},\ \mathbf{A},\ \mathbf{I},\ \mathbf{T},\ \mathbf{R}$ can be written as,

$$\begin{cases}
\mathbf{S}_n(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_1(\vartheta,\mathbf{S}_{n-1}(\vartheta))\Big]d\vartheta,\\
\mathbf{M}_n(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_2(\vartheta,\mathbf{M}_{n-1}(\vartheta))\Big]d\vartheta,\\
\mathbf{E}_n(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_3(\vartheta,\mathbf{E}_{n-1}(\vartheta))\Big]d\vartheta,\\
\mathbf{A}_n(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_4(\vartheta,\mathbf{A}_{n-1}(\vartheta))\Big]d\vartheta,\\
\mathbf{I}_n(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_5(\vartheta,\mathbf{I}_{n-1}(\vartheta))\Big]d\vartheta,\\
\mathbf{T}_n(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_6(\vartheta,\mathbf{T}_{n-1}(\vartheta))\Big]d\vartheta,\\
\mathbf{R}_n(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_7(\vartheta,\mathbf{R}_{n-1}(\vartheta))\Big]d\vartheta,
\end{cases}$$

with initial conditions,

$$\begin{cases}
\mathbf{S}_0(\zeta) = \mathbf{S}(0) \geq 0, \quad \mathbf{M}_0(\zeta) = \mathbf{M}(0) \geq 0, \quad \mathbf{E}_0(\zeta) = \mathbf{E}(0) \geq 0, \quad \mathbf{A}_0(\zeta) = \mathbf{A}(0) \geq 0,\\
\mathbf{I}_0(\zeta) = \mathbf{I}(0) \geq 0, \quad \mathbf{T}_0(\zeta) = \mathbf{T}(0) \geq 0, \quad \mathbf{R}_0(\zeta) = \mathbf{R}(0) \geq 0.
\end{cases}$$

The differences between the successive terms are shown below,

$$\begin{cases}
\Upsilon_{1n}(\zeta) = \mathbf{S}_n(\zeta) - \mathbf{S}_{n-1}(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_1(\vartheta,\mathbf{S}_{n-1}(\vartheta)) - \mathcal{G}_1(\vartheta,\mathbf{S}_{n-2}(\vartheta))\Big]d\vartheta,\\
\Upsilon_{2n}(\zeta) = \mathbf{M}_n(\zeta) - \mathbf{M}_{n-1}(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_2(\vartheta,\mathbf{M}_{n-1}(\vartheta)) - \mathcal{G}_2(\vartheta,\mathbf{M}_{n-2}(\vartheta))\Big]d\vartheta,\\
\Upsilon_{3n}(\zeta) = \mathbf{E}_n(\zeta) - \mathbf{E}_{n-1}(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_3(\vartheta,\mathbf{E}_{n-1}(\vartheta)) - \mathcal{G}_3(\vartheta,\mathbf{E}_{n-2}(\vartheta))\Big]d\vartheta,\\
\Upsilon_{4n}(\zeta) = \mathbf{A}_n(\zeta) - \mathbf{A}_{n-1}(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_4(\vartheta,\mathbf{A}_{n-1}(\vartheta)) - \mathcal{G}_4(\vartheta,\mathbf{A}_{n-2}(\vartheta))\Big]d\vartheta,\\
\Upsilon_{5n}(\zeta) = \mathbf{I}_n(\zeta) - \mathbf{I}_{n-1}(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_5(\vartheta,\mathbf{I}_{n-1}(\vartheta)) - \mathcal{G}_5(\vartheta,\mathbf{I}_{n-2}(\vartheta))\Big]d\vartheta,\\
\Upsilon_{6n}(\zeta) = \mathbf{T}_n(\zeta) - \mathbf{T}_{n-1}(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_6(\vartheta,\mathbf{T}_{n-1}(\vartheta)) - \mathcal{G}_6(\vartheta,\mathbf{T}_{n-2}(\vartheta))\Big]d\vartheta,\\
\Upsilon_{7n}(\zeta) = \mathbf{R}_n(\zeta) - \mathbf{R}_{n-1}(\zeta) = \dfrac{1}{\Gamma(\varrho)}\displaystyle\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_7(\vartheta,\mathbf{R}_{n-1}(\vartheta)) - \mathcal{G}_7(\vartheta,\mathbf{R}_{n-2}(\vartheta))\Big]d\vartheta.
\end{cases} \tag{12}$$

Applying the norm to both sides of (12), and by triangular inequality, we have,

$$\begin{aligned}
\|\Upsilon_{1n}(\zeta)\| = \|\mathbf{S}_n(\zeta) - \mathbf{S}_{n-1}(\zeta)\| &= \left\|\frac{1}{\Gamma(\varrho)}\int_0^\zeta (\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_1(\vartheta,\mathbf{S}_{n-1}(\vartheta)) - \mathcal{G}_1(\vartheta,\mathbf{S}_{n-2}(\vartheta))\Big]d\vartheta\right\|\\
&\leq \frac{1}{\Gamma(\varrho)}\int_0^\zeta \left\|(\zeta-\vartheta)^{\varrho-1}\Big[\mathcal{G}_1(\vartheta,\mathbf{S}_{n-1}(\vartheta)) - \mathcal{G}_1(\vartheta,\mathbf{S}_{n-2}(\vartheta))\Big]\right\|d\vartheta.
\end{aligned}$$

As the kernel compliant the Lipschitz condition, then we have,

$$\|\Upsilon_{1n}(\zeta)\| \leq \frac{\mathcal{H}_1}{\Gamma(\varrho)} \int_0^{\zeta} \|\Upsilon_{1(n-1)}(\vartheta)\| d\vartheta. \tag{13}$$

In the same way, we get,

$$\begin{cases} \|\Upsilon_{2n}(\zeta)\| & \leq \frac{\mathcal{H}_2}{\Gamma(\varrho)} \int_0^{\zeta} \|\Upsilon_{2(n-1)}(\vartheta)\| d\vartheta, \\ \|\Upsilon_{3n}(\zeta)\| & \leq \frac{\mathcal{H}_3}{\Gamma(\varrho)} \int_0^{\zeta} \|\Upsilon_{3(n-1)}(\vartheta)\| d\vartheta, \\ \|\Upsilon_{4n}(\zeta)\| & \leq \frac{\mathcal{H}_4}{\Gamma(\varrho)} \int_0^{\zeta} \|\Upsilon_{4(n-1)}(\vartheta)\| d\vartheta, \\ \|\Upsilon_{5n}(\zeta)\| & \leq \frac{\mathcal{H}_5}{\Gamma(\varrho)} \int_0^{\zeta} \|\Upsilon_{5(n-1)}(\vartheta)\| d\vartheta, \\ \|\Upsilon_{6n}(\zeta)\| & \leq \frac{\mathcal{H}_6}{\Gamma(\varrho)} \int_0^{\zeta} \|\Upsilon_{6(n-1)}(\vartheta)\| d\vartheta, \\ \|\Upsilon_{7n}(\zeta)\| & \leq \frac{\mathcal{H}_7}{\Gamma(\varrho)} \int_0^{\zeta} \|\Upsilon_{7(n-1)}(\vartheta)\| d\vartheta. \end{cases} \tag{14}$$

□

Theorem 4.2. *Let* $\mathcal{W} = \frac{\zeta_0}{\Gamma(\varrho)}\mathcal{H}_i, \ \forall \ i = 1, \ldots, 7$, *then aforementioned Model* (2) *has one solution for finite time* ζ_0, *if* $1 - \mathcal{W} > 0$.

Proof. (13) and (14) are considered and by recursive principle [23], we can write,

$$\|\Upsilon_{in}(\zeta)\| \leq \|\mathcal{H}(0)\| \left[\frac{\mathcal{H}_i}{\Gamma(\varrho)}\right]^n, \ \ i = 1, \ldots, 7. \tag{15}$$

Thus, the proposed Model (2) has at least a solution and also continuous.

Now to show that the above functions are the solutions to the aforementioned Model (2). We consider that,

$$\mathcal{H}(\zeta) - \mathcal{H}(0) = \mathcal{H}_n(\zeta) - \mathcal{F}_{in}(\zeta),$$

where $\mathcal{F}_{in} \ \ (i = 1, \ldots, 7.)$ are remainders terms after n iterations.

By using triangular inequality, we get,

$$\begin{aligned} \|\mathcal{F}_{1n}(\zeta)\| &= \left\| \frac{1}{\Gamma(\varrho)} \int_0^{\zeta} (\zeta - \vartheta)^{\varrho - 1} \Big[\mathcal{G}_1(\vartheta, \mathbf{S}(\vartheta)) - \mathcal{G}_1(\vartheta, \mathbf{S}_{n-1}(\vartheta)) \Big] d\vartheta \right\| \\ &\leq \frac{1}{\Gamma(\varrho)} \int_0^{\zeta} (\zeta - \vartheta)^{\varrho - 1} \Big[\|\mathcal{G}_1(\vartheta, \mathbf{S}(\vartheta)) - \mathcal{G}_1(\vartheta, \mathbf{S}_{n-1}(\vartheta))\| \Big] d\vartheta \\ &\leq \frac{\zeta}{\Gamma(\varrho)} \mathcal{H}_1 \|\mathbf{S} - \mathbf{S}_{n-1}\|. \end{aligned}$$

After using recursion approaches, which gives,

$$\|\mathcal{F}_{1n}(\zeta)\| \leq (\frac{\zeta}{\Gamma(\varrho)} \mathcal{H}_1)^{n+1} c.$$

For $\zeta = \zeta_0$, we have,

$$\|\mathcal{F}_{1n}(\zeta_0)\| \leq (\frac{\zeta_0}{\Gamma(\varrho)}\mathcal{H}_1)^{n+1}c,$$

therefore,

$$\|\mathcal{F}_{1n}(\zeta)\| \to 0, \text{ as } n \to \infty, \text{ if } \frac{\zeta_0}{\Gamma(\varrho)}\mathcal{H}_1 < 1.$$

In the same way, we have,

$$\|\mathcal{F}_{in}(\zeta)\| \to 0, \ as\ n \to \infty, \ i = 2, \ldots, 7.$$

Hence, Model (2) has one solution. □

Theorem 4.3. *If* $(1 - \mathcal{W}) > 0$, *then the Model* (2) *has atmost one unique solution, where,*

$$\mathcal{W} = \frac{\zeta}{\Gamma(\varrho)}\mathcal{H}_i \ \forall \ i = 1, \ldots, 7.$$

Proof. Consider $\mathbf{S}$ and $\mathbf{S}_1$ be the two different solutions of the proposed Model (2), then we get,

$$\mathbf{S}(\zeta) - \mathbf{S}_1(\zeta) = \frac{1}{\Gamma(\varrho)}\int_0^\zeta (\zeta - \vartheta)^{\varrho-1}\Big[\mathcal{G}_1(\vartheta, \mathbf{S}(\vartheta)) - \mathcal{G}_1(\vartheta, \mathbf{S}_1(\vartheta))\Big]d\vartheta. \tag{16}$$

It is clear that,

$$\|\mathbf{S}(\zeta) - \mathbf{S}_1(\zeta)\| \geq 0.$$

Applying norm on (16) and Lipschitz condition of the kernel, we have,

$$\begin{aligned}\|\mathbf{S}(\zeta) - \mathbf{S}_1(\zeta)\| &\leq \frac{1}{\Gamma(\varrho)}\int_0^\zeta (\zeta - \vartheta)^{\varrho-1}\Big[\|\mathcal{G}_1(\vartheta, \mathbf{S}(\vartheta)) - \mathcal{G}_1(\vartheta, \mathbf{S}_1(\vartheta))\|\Big]d\vartheta,\\ &\leq \frac{\zeta}{\Gamma(\varrho)}\mathcal{H}_1\Big[\|\mathbf{S}(\zeta) - \mathbf{S}_1(\zeta)\|\Big],\end{aligned}$$

then,

$$(1 - \frac{\zeta}{\Gamma(\varrho)}\mathcal{H}_1)\Big[\|\mathbf{S}(\zeta) - \mathbf{S}_1(\zeta)\|\Big] \leq 0.$$

Since,

$$(1 - \frac{\zeta}{\Gamma(\varrho)}\mathcal{H}_1) > 0,$$

we have,

$$\|\mathbf{S}(\zeta) - \mathbf{S}_1(\zeta)\| = 0,$$

which give,

$$\mathbf{S}(\zeta) = \mathbf{S}_1(\zeta).$$

In the similar way , we obtain,

$$\mathbf{M}(\zeta) = \mathbf{M}_1(\zeta),\ \mathbf{E}(\zeta) = \mathbf{E}_1(\zeta),\ \mathbf{A}(\zeta) = \mathbf{A}_1(\zeta),\ \mathbf{I}(\zeta) = \mathbf{I}_1(\zeta),\ \mathbf{T}(\zeta) = \mathbf{T}_1(\zeta),\ \mathbf{R}(\zeta) = \mathbf{R}_1(\zeta).$$

Thus, the Model (2) has atmost one unique solution. □

5 Fundamental Characteristics

5.1 Invariant region

Let $\Omega = \{(\mathbf{S}, \mathbf{M}, \mathbf{E}, \mathbf{A}, \mathbf{I}, \mathbf{T}, \mathbf{R}) \in R_+^7\}$ be the feasible region, and let the functions on the right side of Model (2) be continuous on R_+^7. The net population is obtained by adding the all classes.

$$^{C}D^{\varrho}\mathbf{N}(\zeta) = {}^{C}D^{\varrho}\mathbf{S} + {}^{C}D^{\varrho}\mathbf{M} + {}^{C}D^{\varrho}\mathbf{E} + {}^{C}D^{\varrho}\mathbf{A} + {}^{C}D^{\varrho}\mathbf{I} + {}^{C}D^{\varrho}\mathbf{T} + {}^{C}D^{\varrho}\mathbf{R},$$

which gives,

$$^{C}D^{\varrho}\mathbf{N}(\zeta) + \eta^{\varrho}\mathbf{N}(\zeta) \leq \Lambda^{\varrho}. \tag{17}$$

Using (17) and the Laplace transform, we obtain,

$$\mathbf{N}(\zeta) \leq \frac{\Lambda^{\varrho}}{\eta^{\varrho}}, \ \forall\, \zeta.$$

Thus, the solutions of the Model (2) restricted to the domain Ω. Hence, Ω is positively invariant.

5.2 Positivity and boundedness

To demonstrate that all solutions to the aforementioned Model (2) are positive, we notice that,

$$\begin{cases} {}^{C}D_0^{\varrho}\mathbf{S} = \Lambda^{\varrho} + \rho^{\varrho}\mathbf{M} + \lambda^{\varrho}\mathbf{E} + \sigma^{\varrho}\mathbf{R} > 0, \\ {}^{C}D_0^{\varrho}\mathbf{M} = \pi^{\varrho}\Lambda^{\varrho} > 0, \\ {}^{C}D_0^{\varrho}\mathbf{E} = \alpha^{\varrho}\mathbf{S}(\mathbf{I} + \beta^{\varrho}\mathbf{T}) \geq 0, \\ {}^{C}D_0^{\varrho}\mathbf{A} = r_1\lambda^{\varrho}\mathbf{E} \geq 0, \\ {}^{C}D_0^{\varrho}\mathbf{I} = \omega^{\varrho}\mathbf{E} + \theta^{\varrho}\mathbf{A} \geq 0, \\ {}^{C}D_0^{\varrho}\mathbf{T} = \xi^{\varrho}\mathbf{I} \geq 0, \\ {}^{C}D_0^{\varrho}\mathbf{R} = \gamma^{\varrho}\mathbf{E} + \mu^{\varrho}\mathbf{T} \geq 0. \end{cases}$$

According to Corollary 3.1, the result is in R_+^7 i.e.,

$$\Omega = \{(\mathbf{S}, \mathbf{M}, \mathbf{E}, \mathbf{A}, \mathbf{I}, \mathbf{T}, \mathbf{R}) \in R_+^7 \mid (\mathbf{S} + \mathbf{M} + \mathbf{E} + \mathbf{A} + \mathbf{I} + \mathbf{T} + \mathbf{R}) \geq 0\}.$$

Thus, the Model (2) solutions are positive and bounded in the aforementioned feasible region Ω.

5.3 Equilibrium points of the Model (2)

In this section, we will compute the equilibrium points of Model (2). The equilibrium points for the system are derived as,

$$\begin{cases} (1-\pi^{\varrho})\Lambda^{\varrho} + \rho^{\varrho}\mathbf{M} + (1-r_1)\lambda^{\varrho}\mathbf{E} + \sigma^{\varrho}\mathbf{R} - \alpha^{\varrho}\mathbf{S}(\mathbf{I} + \beta^{\varrho}\mathbf{T}) - \eta^{\varrho}\mathbf{S} = 0, \\ \pi^{\varrho}\Lambda^{\varrho} - k_1\mathbf{M} = 0, \\ \alpha^{\varrho}\mathbf{S}(\mathbf{I} + \beta^{\varrho}\mathbf{T}) - k_2\mathbf{E} = 0, \\ r_1\lambda^{\varrho}\mathbf{E} - k_3\mathbf{A} = 0, \\ \omega^{\varrho}\mathbf{E} + \theta^{\varrho}\mathbf{A} - k_4\mathbf{I} = 0, \\ \xi^{\varrho}\mathbf{I} - k_5\mathbf{T} = 0, \\ \gamma^{\varrho}\mathbf{E} + \mu^{\varrho}\mathbf{T} - k_6\mathbf{R} = 0. \end{cases} \tag{18}$$

Put $\mathbf{E} = \mathbf{I} = \mathbf{T} = \mathbf{A} = 0$ in (18), after simplification we have,

$$\mathbf{E^o} = \left(\frac{(k_1(1-\pi^\varrho) + \rho^\varrho \pi^\varrho)\Lambda^\varrho}{\eta^\varrho k_1}, \frac{\pi^\varrho \Lambda^\varrho}{k_1}, 0, 0, 0, 0 \right).$$

A disease endemic equilibrium $\mathbf{E^e}$ exists. Let $\mathbf{S} = \mathbf{S}^*$, $\mathbf{M} = \mathbf{M}^*$, $\mathbf{E} = \mathbf{E}^*$, $\mathbf{A} = \mathbf{A}^*$, $\mathbf{I} = \mathbf{I}^*$, $\mathbf{T} = \mathbf{T}^*$ and $\mathbf{R} = \mathbf{R}^*$ in (18), then we have,

$$\begin{cases} (1-\pi^\varrho)\Lambda^\varrho + \rho^\varrho \mathbf{M}^* + (1-r_1)\lambda^\varrho \mathbf{E}^* + \sigma^\varrho \mathbf{R}^* - \alpha^\varrho \mathbf{S}^*(\mathbf{I}^* + \beta^\varrho \mathbf{T}^*) - \eta^\varrho \mathbf{S}^* = 0, \\ \pi^\varrho \Lambda^\varrho - k_1 \mathbf{M}^* = 0, \\ \alpha^\varrho \mathbf{S}^*(\mathbf{I}^* + \beta^\varrho \mathbf{T}^*) - k_2 \mathbf{E}^* = 0, \\ r_1 \lambda^\varrho \mathbf{E}^* - k_3 \mathbf{A}^* = 0, \\ \omega^\varrho \mathbf{E}^* + \theta^\varrho \mathbf{A}^* - k_4 \mathbf{I}^* = 0, \\ \xi^\varrho \mathbf{I}^* - k_5 \mathbf{T}^* = 0, \\ \gamma^\varrho \mathbf{E}^* + \mu^\varrho \mathbf{T}^* - k_6 \mathbf{R}^* = 0. \end{cases}$$

After simplify, we get,

$$\mathbf{S}^* = \frac{k_2 k_3 k_4 k_5}{\Pi}, \qquad \mathbf{M}^* = \frac{\pi^\varrho \Lambda^\varrho}{k_1},$$
$$\mathbf{A}^* = \frac{\eta^\varrho r_1 \lambda^\varrho k_2 k_3 k_4^2 k_5^2 k_6 (\mathbf{R}_0 - 1)}{\Pi\Omega}, \qquad \mathbf{I}^* = \frac{\eta^\varrho k_2 k_3 k_4 k_5^2 k_6 (\mathbf{R}_0 - 1)}{\Pi\Omega},$$
$$\mathbf{T}^* = \frac{\eta^\varrho \xi^\varrho k_2 k_3 k_4 k_5 k_6 (\mathbf{R}_0 - 1)}{\Pi\Omega}, \qquad \mathbf{E}^* = \frac{\eta^\varrho k_2 k_3^2 k_4^2 k_5^2 k_6 (\mathbf{R}_0 - 1)}{\Pi\Omega},$$
$$\mathbf{R}^* = \frac{\eta^\varrho k_2 k_3 k_4 k_5 (\gamma^\varrho k_3 k_4 k_5 + \nu^\varrho \xi^\varrho (\omega^\varrho k_3 + \theta^\varrho r_1 \lambda^\varrho))(\mathbf{R}_0 - 1)}{\Pi\Omega},$$

where,

$$\Omega = (\lambda^\varrho(1-r_1) - k_2) k_3 k_4 k_5 k_6 + \sigma^\varrho(\gamma^\varrho k_3 k_4 k_5 + \mu^\varrho \xi^\varrho (\omega^\varrho k_3 + \theta^\varrho r_1 \lambda^\varrho)),$$
$$\Pi = \alpha^\varrho (k_5 + \beta^\varrho \xi^\varrho)(\omega^\varrho k_3 + \theta^\varrho r_1 \lambda^\varrho).$$

5.3.1 Basic reproduction number $\mathbf{R}_0$

The given Model (2) $\mathbf{R}_0$ is calculated using the next–generation matrix technique [42]. Let the infected system $(\mathbf{E}, \mathbf{A}, \mathbf{I}, \mathbf{T})^T$, then the Jacobian matrices computed at $\mathbf{E_0}$ are given as,

$$\mathsf{F}(\mathbf{E_0}) = \begin{bmatrix} 0 & 0 & \frac{(k_1(1-\pi^\varrho) + \rho^\varrho \pi^\varrho)\alpha^\varrho \Lambda^\varrho}{\eta^\varrho k_1} & \frac{(k_1(1-\pi^\varrho) + \rho^\varrho \pi^\varrho)\alpha^\varrho \beta^\varrho \Lambda^\varrho}{\eta^\varrho k_1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathsf{V} = \begin{bmatrix} k_2 & 0 & 0 & 0 \\ -r_1\lambda^\varrho & k_3 & 0 & 0 \\ -\omega^\varrho & -\theta^\varrho & k_4 & 0 \\ 0 & 0 & -\xi^\varrho & k_5 \end{bmatrix}.$$

After simplifying $\mathsf{F}(\mathbf{E_0})\mathsf{V}^{-1}$, we get,

$$\mathbf{R}_0 = \frac{(k_1 - \pi^\varrho k_1 + \rho^\varrho \pi^\varrho)(k_5 + \beta^\varrho \xi^\varrho)\alpha^\varrho(\omega^\varrho k_3 + \theta^\varrho \lambda^\varrho r_1)\Lambda^\varrho}{\eta^\varrho k_1 k_2 k_3 k_4 k_5}.$$

6 Hyers–Ulam Stability

The Hyers–Ulam stability analysis of fractional–order systems is a critical aspect of ensuring that mathematical models behave reliably, even under small perturbations. This is especially important in biological and epidemiological applications, where accurate predictions of disease dynamics are essential for effective public health interventions. By ensuring the stability of fractional–order models, such as those used in TB transmission, researchers can increase confidence in their models' predictions and provide more reliable recommendations for disease control strategies. This section examines the Hyers–Ulam stability of the model (2). The necessary inequalities are defined below.

Definition 6.1. *If there are constants* $\mathcal{K}_i > 0,\ i \in \mathsf{N}_1^7$ *satisfying; for every* $\epsilon_i > 0,\ i \in \mathsf{N}_1^7$,

$$\begin{aligned}
\left|{}^{C}D_0^{\varrho}\mathbf{S}(\zeta) - \mathcal{G}_1(\zeta,\mathbf{S})\right| &\leq \epsilon_1,\\
\left|{}^{C}D_0^{\varrho}\mathbf{M}(\zeta) - \mathcal{G}_2(\zeta,\mathbf{M})\right| &\leq \epsilon_2,\\
\left|{}^{C}D_0^{\varrho}\mathbf{E}(\zeta) - \mathcal{G}_3(\zeta,\mathbf{E})\right| &\leq \epsilon_3,\\
\left|{}^{C}D_0^{\varrho}\mathbf{A}(\zeta) - \mathcal{G}_4(\zeta,\mathbf{A})\right| &\leq \epsilon_4,\\
\left|{}^{C}D_0^{\varrho}\mathbf{I}(\zeta) - \mathcal{G}_5(\zeta,\mathbf{I})\right| &\leq \epsilon_5,\\
\left|{}^{C}D_0^{\varrho}\mathbf{T}(\zeta) - \mathcal{G}_6(\zeta,\mathbf{T})\right| &\leq \epsilon_6,\\
\left|{}^{C}D_0^{\varrho}\mathbf{R}(\zeta) - \mathcal{G}_7(\zeta,\mathbf{R})\right| &\leq \epsilon_7,
\end{aligned} \tag{19}$$

and TB Model (2) *posses a solution* $(\mathbf{S},\mathbf{M},\mathbf{E},\mathbf{A},\mathbf{I},\mathbf{T},\mathbf{R})$ *satisfying* $\|S-\mathbf{S}\| \leq \mathcal{K}_1\epsilon_1$, $\|M-\mathbf{M}\| \leq \mathcal{K}_2\epsilon_2$, $\|E-\mathbf{E}\| \leq \mathcal{K}_3\epsilon_3$, $\|A-\mathbf{A}\| \leq \mathcal{K}_4\epsilon_4$, $\|I-\mathbf{I}\| \leq \mathcal{K}_5\epsilon_5$, $\|T-\mathbf{T}\| \leq \mathcal{K}_6\epsilon_6$, $\|R-\mathbf{R}\| \leq \mathcal{K}_7\epsilon_7$, *then proposed Model* (2) *is Hyers–Ulam stable.*

Remark 6.1. *Consider* $\mathbf{S}(\zeta),\ \mathbf{M}(\zeta),\ \mathbf{E}(\zeta),\ \mathbf{A}(\zeta),\mathbf{I}(\zeta),\ \mathbf{T}(\zeta)$, *and* $\mathbf{R}(\zeta)$ *are the solutions of inequalities* (19), *and if there are* $h_i,\ i = 1,\ldots,7$, *such that* $|h_i(\zeta)| < \epsilon_i$ *we have,*

$$\begin{aligned}
{}^{C}D_0^{\varrho}\mathbf{S}(\zeta) &= \mathcal{G}_1(\zeta,\mathbf{S}) + h_1(\zeta),\\
{}^{C}D_0^{\varrho}\mathbf{M}(\zeta) &= \mathcal{G}_2(\zeta,\mathbf{M}) + h_2(\zeta),\\
{}^{C}D_0^{\varrho}\mathbf{E}(\zeta) &= \mathcal{G}_3(\zeta,\mathbf{E}) + h_3(\zeta),\\
{}^{C}D_0^{\varrho}\mathbf{A}(\zeta) &= \mathcal{G}_4(\zeta,\mathbf{A}) + h_4(\zeta),\\
{}^{C}D_0^{\varrho}\mathbf{I}(\zeta) &= \mathcal{G}_5(\zeta,\mathbf{I}) + h_5(\zeta),\\
{}^{C}D_0^{\varrho}\mathbf{T}(\zeta) &= \mathcal{G}_6(\zeta,\mathbf{T}) + h_6(\zeta),\\
{}^{C}D_0^{\varrho}\mathbf{R}(\zeta) &= \mathcal{G}_7(\zeta,\mathbf{R}) + h_7(\zeta).
\end{aligned} \tag{20}$$

Theorem 6.1. *Let* $\dfrac{\zeta}{\Gamma(\varrho)}\gamma_i < 1,\ \forall\ i = 1,\ldots,7$, *holds. Then the proposed TB Model* (2) *is Hyers–Ulam stable.*

Proof. Let $\epsilon_1 > 0$ and the function $\mathbf{S}$ be continuous such that,

$$\left|{}^{C}D_0^{\varrho}\mathbf{S}(\zeta) - \mathcal{G}_1(\zeta,\mathbf{S})\right| \leq \epsilon_1.$$

From Remark 6.1, we obtain,

$$^{C}D_0^{\varrho}\mathbf{S}(\zeta) = \mathcal{G}_1(\zeta, \mathbf{S}) + h_1(\zeta).$$

Consequently,

$$\mathbf{S}(\zeta) = \mathbf{S}(0) + \frac{1}{\Gamma(\varrho)}\int_0^{\zeta}(\zeta-\vartheta)^{\varrho-1}[\mathcal{G}_1(\vartheta, \mathbf{S}(\vartheta))]d\vartheta + \frac{1}{\Gamma(\varrho)}\int_0^{\zeta}(\zeta-\vartheta)^{\varrho-1}[h_1(\vartheta)]d\vartheta.$$

Assume that $\mathbf{S}_1(\zeta)$ be a unique solution of the model, then we have,

$$\mathbf{S}_1(\zeta) = \mathbf{S}_1(0) + \frac{1}{\Gamma(\varrho)}\int_0^{\zeta}(\zeta-\vartheta)^{\varrho-1}[\mathcal{G}_1(\vartheta, \mathbf{S}_1(\vartheta))]d\vartheta,$$

and,

$$\begin{aligned}\left\|\mathbf{S}-\mathbf{S}_1\right\| &\leq \frac{1}{\Gamma(\varrho)}\int_0^{\zeta}(\zeta-\vartheta)^{\varrho-1}\left\|\mathcal{G}_1(\vartheta, \mathbf{S}(\vartheta)) - \mathcal{G}_1(\vartheta, \mathbf{S}_1(\vartheta))\right\|d\vartheta + \frac{1}{\Gamma(\varrho)}\int_0^{\zeta}(\zeta-\vartheta)^{\varrho-1}\left\|h_1(\vartheta)\right\|d\vartheta,\\ &\leq \frac{\zeta}{\Gamma(\varrho)}\gamma_1\left\|\mathbf{S}-\mathbf{S}_1\right\| + \frac{\zeta}{\Gamma(\varrho)}\epsilon_1.\end{aligned}$$

Thus,

$$\|\mathbf{S}-\mathbf{S}_1\| \leq \frac{\left[\frac{\zeta}{\Gamma(\varrho)}\right]}{\left[1-\frac{\zeta\gamma_1}{\Gamma(\varrho)}\right]}\epsilon_1,$$

then,

$$\|\mathbf{S}-\mathbf{S}_1\| \leq \mathcal{K}_1\epsilon_1,$$

where $\mathcal{K}_1 = \dfrac{\left[\frac{\zeta}{\Gamma(\varrho)}\right]}{\left[1-\frac{\zeta\gamma_1}{\Gamma(\varrho)}\right]}$.

Similarly,

$$\begin{aligned}&\|\mathbf{M}-\mathbf{M}_1\| \leq \mathcal{K}_2\epsilon_2, \quad &&\|\mathbf{E}-\mathbf{E}_1\| \leq \mathcal{K}_3\epsilon_3, \quad &&\|\mathbf{A}-\mathbf{A}_1\| \leq \mathcal{K}_4\epsilon_4,\\ &\|\mathbf{I}-\mathbf{I}_1\| \leq \mathcal{K}_5\epsilon_5, \quad &&\|\mathbf{T}-\mathbf{T}_1\| \leq \mathcal{K}_6\epsilon_6, \quad &&\|\mathbf{R}-\mathbf{R}_1\| \leq \mathcal{K}_7\epsilon_7,\end{aligned}$$

where $\mathcal{K}_i = \dfrac{\left[\frac{\zeta}{\Gamma(\varrho)}\right]}{\left[1-\frac{\zeta\gamma_i}{\Gamma(\varrho)}\right]}$, $i = 2, \ldots, 7$.

Hence, proposed Model (2) is Hyers–Ulam stable. □

7 Sensitivity Analysis

This study examines how changing parameters affect $\mathbf{R}_0$ and the potential for disease control and elimination. To calculate the partial derivatives of $\mathbf{R}_0$, use Definition 3.2 with regard to the

parameters π, μ, ψ_1, ψ_2, ξ, γ, λ, ρ, θ, ω, α, Λ and β. A positive (negative) index indicates that an increase in the parameter value causes a rise (reduction) in $\mathbf{R}_0$ [38].

The sensitivity index $\mathbf{R}_0$ for the model parameters is calculated by,

$$\Psi_{\xi^\varrho}^{\mathbf{R}_0} = -\frac{\alpha^\varrho\Lambda^\varrho AC(B-k_4\beta^\varrho)}{k_4\eta^\varrho E} < 0, \quad \Psi_{\theta^\varrho}^{\mathbf{R}_0} = -\frac{\alpha^\varrho\Lambda^\varrho AB(C-\lambda^\varrho r_1 k_3)}{k_3\eta^\varrho E} < 0, \quad \Psi_{\beta^\varrho}^{\mathbf{R}_0} = \frac{\alpha^\varrho\xi^\varrho\Lambda^\varrho AC}{\eta^\varrho E} > 0,$$

$$\Psi_{\psi_2^\varrho}^{\mathbf{R}_0} = -\frac{\alpha^\varrho\Lambda^\varrho AC(k_5-B)}{k_5\eta^\varrho E} < 0, \quad \Psi_{\mu^\varrho}^{\mathbf{R}_0} = -\frac{\alpha^\varrho\Lambda^\varrho AC(k_5-B)}{k_5\eta^\varrho E} < 0, \quad \Psi_{\gamma^\varrho}^{\mathbf{R}_0} = -\frac{\alpha^\varrho\Lambda^\varrho ABC}{k_2\eta^\varrho E} < 0,$$

$$\Psi_{\lambda^\varrho}^{\mathbf{R}_0} = \frac{\alpha^\varrho\Lambda^\varrho AB(\theta^\varrho r_1 k_2-C)}{k_2\eta^\varrho E} > 0, \quad \Psi_{\rho^\varrho}^{\mathbf{R}_0} = \frac{\alpha^\varrho\Lambda^\varrho BC(k_1-A)}{k_1\eta^\varrho E} > 0, \quad \Psi_{\alpha^\varrho}^{\mathbf{R}_0} = \frac{\Lambda^\varrho ABC}{\eta^\varrho E} > 0,$$

$$\Psi_{\pi^\varrho}^{\mathbf{R}_0} = -\frac{\alpha^\varrho\Lambda^\varrho BC(k_1-\rho^\varrho)}{\eta^\varrho E} < 0, \quad \Psi_{\Lambda^\varrho}^{\mathbf{R}_0} = \frac{\alpha^\varrho ABC}{k_2\eta^\varrho E} > 0, \quad \Psi_{\psi_1^\varrho}^{\mathbf{R}_0} = -\frac{\alpha^\varrho\Lambda^\varrho ABC}{k_4\eta^\varrho E} < 0,$$

$$\Psi_{\eta^\varrho}^{\mathbf{R}_0} = -\alpha^\varrho\Lambda^\varrho\left[\frac{ABC(E+\eta(k_3k_4k_5(k_1+k_2)+k_1k_2(k_3k_4+k_3k_5+k_4k_5)))}{\eta^\varrho E^2} - \frac{(A(\omega^\varrho B+C)+(1-\pi)BC))}{\eta^\varrho E}\right] < 0, \quad \Psi_{\omega^\varrho}^{\mathbf{R}_0} = \frac{\alpha^\varrho\Lambda^\varrho AB(k_3k_2-C)}{\eta^\varrho E} > 0,$$

where $A = k_1 - \pi^\varrho k_1 + \rho^\varrho\pi^\varrho$, $B = k_5 + \beta^\varrho\xi^\varrho$, $C = \omega^\varrho k_3 + \theta^\varrho\lambda^\varrho r_1$, and $E = k_1k_2k_3k_4k_5$.

7.1 Impact of θ on the infected population class (I)

Figure 2 shows that the effects on the TB disease through parameters variation. The obtained graph presents the number of infected individuals using the parameter value (dot line) described in Table 1 and the corresponding curves with a specific parameter value increases of 10% (solid line), respectively.

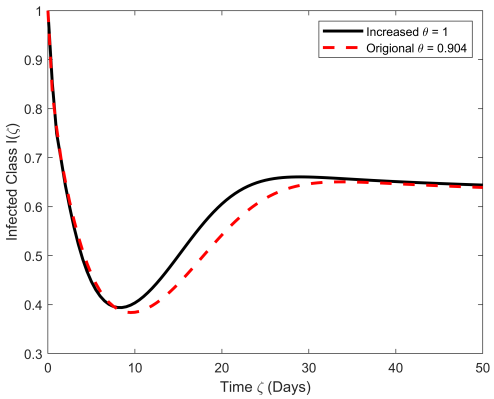


Figure 2: Effects of θ on the infected class (**I**).

It means that the small changes from 0.904 to 1 to the value of θ, resulting the small increment in the infected population. After 50 days, both graphs shows stability and convergence.

Table 1: Parameters values of the proposed model.

Parameter	Description	Value	Source
π	Rate of immunized at birth	0.065	[32]
Λ	Rate of procurement	1	[32]
ρ	Fraction of weaning off the medication	0.0256	[32]
η	Natural death rate	0.021	[32]
β	Rate of infection person from $\mathbf{I}$ enter into $\mathbf{T}$	0.1	[3]
r_1	Rate of progression from $\mathbf{E}$ enter into $\mathbf{A}$	0.23	[3]
λ	Latency or incubation period	0.811	[32]
θ	Rate of individuals from $\mathbf{A}$ to $\mathbf{I}$	0.904	[32]
α	Rate of effective contact	0.7	[3]
σ	Rate of recovered class being suspected	0.5	assumed
γ	Rate of therapeutic efficacy of contagious predisposition	0.0342	[32]
ω	Rate of collapse of innate TB into extremely contagious TB	0.124	[32]
ξ	Rate of effective remedy of TB patients $\mathbf{I}$	0.2	[3]
ψ_1	Disease induced death rate in $\mathbf{I}$	0.15	[3]
ψ_2	Disease induced death rate in $\mathbf{T}$	0.05	[3]
μ	Rate of individuals from $\mathbf{T}$ enter into $\mathbf{R}$	0.1	[32]
N_1		7	
N_2		5	
N_3		2	
N_4		1	
N_5		1	
N_6		0	
N_7		0	

8 Numerical Analysis

8.1 The construction of solution for Model (2) by HPM

Apply the HPM [24] on the Model (2) yields;

$$\begin{cases} (1-\mathcal{H}_\mathcal{P})[{}^C D^\varrho \mathbf{S}(\zeta) - {}^C D^\varrho \mathbf{S}_0(\zeta)] + \mathcal{H}_\mathcal{P}[{}^C D^\varrho \mathbf{S}(\zeta)] = \mathcal{H}_\mathcal{P}[(1-\pi^\varrho)\Lambda^\varrho + \rho^\varrho \mathbf{M} \\ \quad +(1-r_1)\lambda^\varrho \mathbf{E} + \sigma^\varrho \mathbf{R} - \alpha^\varrho \mathbf{S}(\mathbf{I}+\beta^\varrho \mathbf{T}) - \eta^\varrho \mathbf{S}], \\ (1-\mathcal{H}_\mathcal{P})[{}^C D^\varrho \mathbf{M}(\zeta) - {}^C D^\varrho \mathbf{M}_0(\zeta)] + \mathcal{H}_\mathcal{P}[{}^C D^\varrho \mathbf{M}(\zeta)] = \mathcal{H}_\mathcal{P}[\pi^\varrho \Lambda^\varrho - k_1 \mathbf{M}], \\ (1-\mathcal{H}_\mathcal{P})[{}^C D^\varrho \mathbf{E}(\zeta) - {}^C D^\varrho \mathbf{E}_0(\zeta)] + \mathcal{H}_\mathcal{P}[{}^C D^\varrho \mathbf{E}_(\zeta)] = \mathcal{H}_\mathcal{P}[\alpha^\varrho \mathbf{S}(\mathbf{I}+\beta^\varrho \mathbf{T}) - k_2 \mathbf{E}], \\ (1-\mathcal{H}_\mathcal{P})[{}^C D^\varrho \mathbf{A}(\zeta) - {}^C D^\varrho \mathbf{A}_0(\zeta)] + \mathcal{H}_\mathcal{P}[{}^C D^\varrho \mathbf{A}_(\zeta)] = \mathcal{H}_\mathcal{P}[r_1 \lambda^\varrho \mathbf{E} - k_3 \mathbf{A}], \\ (1-\mathcal{H}_\mathcal{P})[{}^C D^\varrho \mathbf{I}(\zeta) - {}^C D^\varrho \mathbf{I}_0(\zeta)] + \mathcal{H}_\mathcal{P}[{}^C D^\varrho \mathbf{I}(\zeta)] = \mathcal{H}_\mathcal{P}[\omega^\varrho \mathbf{E} + \theta^\varrho \mathbf{A} - k_4 \mathbf{I}], \\ (1-\mathcal{H}_\mathcal{P})[{}^C D^\varrho \mathbf{T}(\zeta) - {}^C D^\varrho \mathbf{T}_0(\zeta)] + \mathcal{H}_\mathcal{P}[{}^C D^\varrho \mathbf{T}(\zeta)] = \mathcal{H}_\mathcal{P}[\xi^\varrho \mathbf{I} - k_5 \mathbf{T}], \\ (1-\mathcal{H}_\mathcal{P})[{}^C D^\varrho \mathbf{R}(\zeta) - {}^C D^\varrho \mathbf{R}_0(\zeta)] + \mathcal{H}_\mathcal{P}[{}^C D^\varrho \mathbf{R}(\zeta)] = \mathcal{H}_\mathcal{P}[\gamma^\varrho \mathbf{E} + \mu^\varrho \mathbf{T} - k_6 \mathbf{R}]. \end{cases} \tag{21}$$

Let,

$$\begin{cases} \mathbf{S} = \mathbf{S}_0 + \mathcal{H}_\mathcal{P}\mathbf{S}_1 + {\mathcal{H}_\mathcal{P}}^2\mathbf{S}_2 + \cdots \\ \mathbf{M} = \mathbf{M}_0 + \mathcal{H}_\mathcal{P}\mathbf{M}_1 + {\mathcal{H}_\mathcal{P}}^2\mathbf{M}_2 + \cdots \\ \mathbf{E} = \mathbf{E}_0 + \mathcal{H}_\mathcal{P}\mathbf{E}_1 + {\mathcal{H}_\mathcal{P}}^2\mathbf{E}_2 + \cdots \\ \mathbf{A} = \mathbf{A}_0 + \mathcal{H}_\mathcal{P}\mathbf{A}_1 + {\mathcal{H}_\mathcal{P}}^2\mathbf{A}_2 + \cdots \\ \mathbf{I} = \mathbf{I}_0 + \mathcal{H}_\mathcal{P}\mathbf{I}_1 + {\mathcal{H}_\mathcal{P}}^2\mathbf{I}_2 + \cdots \\ \mathbf{T} = \mathbf{T}_0 + \mathcal{H}_\mathcal{P}\mathbf{T}_1 + {\mathcal{H}_\mathcal{P}}^2\mathbf{T}_2 + \cdots \\ \mathbf{R} = \mathbf{R}_0 + \mathcal{H}_\mathcal{P}\mathbf{R}_1 + {\mathcal{H}_\mathcal{P}}^2\mathbf{R}_2 + \cdots \end{cases} \tag{22}$$

Substitute $\mathcal{H}_\mathcal{P} = 0$ in the aforementioned system (21). Then we get,

$$\begin{cases} {}^C D^\varrho \mathbf{S}(\zeta) - {}^C D^\varrho \mathbf{S}_0(\zeta) = 0, \\ {}^C D^\varrho \mathbf{M}(\zeta) - {}^C D^\varrho \mathbf{M}_0(\zeta) = 0, \\ {}^C D^\varrho \mathbf{E}(\zeta) - {}^C D^\varrho \mathbf{E}_0(\zeta) = 0, \\ {}^C D^\varrho \mathbf{A}(\zeta) - {}^C D^\varrho \mathbf{A}_0(\zeta) = 0, \\ {}^C D^\varrho \mathbf{I}(\zeta) - {}^C D^\varrho \mathbf{I}_0(\zeta) = 0, \\ {}^C D^\varrho \mathbf{T}(\zeta) - {}^C D^\varrho \mathbf{T}_0(\zeta) = 0, \\ {}^C D^\varrho \mathbf{R}(\zeta) - {}^C D^\varrho \mathbf{R}_0(\zeta) = 0. \end{cases} \tag{23}$$

Substitute (23) into (21) and comparing terms with the power of $\mathcal{H}_\mathcal{P}$ yields;

$$\begin{aligned} \mathbf{S}_0(\zeta) &= N_1, & \mathbf{M}_0(\zeta) &= N_2, & \mathbf{E}_0(\zeta) &= N_3, & \mathbf{A}_0(\zeta) &= N_4, \\ \mathbf{I}_0(\zeta) &= N_5, & \mathbf{T}_0(\zeta) &= N_6, & \mathbf{R}_0(\zeta) &= N_7. \end{aligned}$$

Similarly,

$$\begin{cases} \mathbf{S}_1(\zeta) = [(1-\pi^\varrho)\Lambda^\varrho + \rho^\varrho N_2 + (1-r_1)\lambda^\varrho N_3 + \sigma^\varrho N_7 - \alpha^\varrho N_1(N_5 + \beta^\varrho N_6) - \eta^\varrho N_1]\dfrac{\zeta^\varrho}{\Gamma(\varrho+1)}, \\ \mathbf{M}_1(\zeta) = [\pi^\varrho \Lambda^\varrho - k_1 N_2]\dfrac{\zeta^\varrho}{\Gamma(\varrho+1)}, \\ \mathbf{E}_1(\zeta) = [\alpha^\varrho N_1(N_5 + \beta^\varrho N_6) - k_2 N_3]\dfrac{\zeta^\varrho}{\Gamma(\varrho+1)}, \\ \mathbf{A}_1(\zeta) = [r_1\lambda^\varrho N_3 - k_3 N_4]\dfrac{\zeta^\varrho}{\Gamma(\varrho+1)}, \\ \mathbf{I}_1(\zeta) = [\omega^\varrho N_3 + \theta^\varrho N_4 - k_4 N_5]\dfrac{\zeta^\varrho}{\Gamma(\varrho+1)}, \\ \mathbf{T}_1(\zeta) = [\xi^\varrho N_5 - k_5 N_6]\dfrac{\zeta^\varrho}{\Gamma(\varrho+1)}, \\ \mathbf{R}_1(\zeta) = [\gamma^\varrho N_3 + \mu^\varrho N_6 - k_6 N_7]\dfrac{\zeta^\varrho}{\Gamma(\varrho+1)}. \end{cases}$$

$$\begin{cases} \mathbf{S}_2(\zeta) = \dfrac{(1-\pi^{\varrho})\Lambda^{\varrho}\zeta^{\varrho}}{\Gamma(\varrho+1)} + [\rho^{\varrho}v_{11} + (1-r_1)\lambda^{\varrho}w_{11} + \sigma^{\varrho}a_{11} - \alpha^{\varrho}(N_1(y_{11}+\beta^{\varrho}z_{11}) + u_{11}N_5 \\ \qquad + \beta^{\varrho}u_{11}N_6) - \eta^{\varrho}u_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)}, \\ \mathbf{M}_2(\zeta) = \dfrac{\pi^{\varrho}\Lambda^{\varrho}\zeta^{\varrho}}{\Gamma(\varrho+1)} - [k_1v_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)}, \\ \mathbf{E}_2(\zeta) = [\alpha^{\varrho}(N_1(y_{11}+\beta^{\varrho}z_{11}) + u_{11}N_5 + \beta^{\varrho}u_{11}N_6) - k_2w_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)}, \\ \mathbf{A}_2(\zeta) = [r_1\lambda^{\varrho}w_{11} - k_3x_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)}, \\ \mathbf{I}_2(\zeta) = [\omega^{\varrho}w_{11} + \theta^{\varrho}x_{11} - k_4y_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)}, \\ \mathbf{T}_2(\zeta) = [\xi^{\varrho}y_{11} - k_5z_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)}, \\ \mathbf{R}_2(\zeta) = [\gamma^{\varrho}w_{11} + \mu^{\varrho}z_{11} - k_6a_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)}. \end{cases}$$

After substitution $\mathcal{H}_{\mathcal{P}} = 1$ in (22), we get,

$$\begin{cases} \mathbf{S}(\zeta) = \mathbf{S}_0 + [(1-\pi^{\varrho})\Lambda^{\varrho} + u_{11}]\dfrac{\zeta^{\varrho}}{\Gamma(\varrho+1)} + [\rho^{\varrho}v_{11} + (1-r_1)\lambda^{\varrho}w_{11} + \sigma^{\varrho}a_{11} \\ \qquad - \alpha^{\varrho}(N_1(y_{11}+\beta^{\varrho}z_{11}) + u_{11}N_5 + \beta^{\varrho}u_{11}N_6) - \eta^{\varrho}u_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)} + \dots \\ \mathbf{M}(\zeta) = \mathbf{M}_0 + [\pi^{\varrho}\Lambda^{\varrho} + v_{11}]\dfrac{\zeta^{\varrho}}{\Gamma(\varrho+1)} - [k_1v_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)} + \dots \\ \mathbf{E}(\zeta) = \mathbf{E}_0 + \dfrac{w_{11}\zeta^{\varrho}}{\Gamma(\varrho+1)} + [\alpha^{\varrho}(N_1(y_{11}+\beta^{\varrho}z_{11}) + u_{11}N_5 + \beta^{\varrho}u_{11}N_6) - k_2w_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)} + \dots \\ \mathbf{A}(\zeta) = \mathbf{A}_0 + \dfrac{x_{11}\zeta^{\varrho}}{\Gamma(\varrho+1)} + [r_1\lambda^{\varrho}w_{11} - k_3x_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)} + \dots \\ \mathbf{I}(\zeta) = \mathbf{I}_0 + \dfrac{y_{11}\zeta^{\varrho}}{\Gamma(\varrho+1)} + [\omega^{\varrho}w_{11} + \theta^{\varrho}x_{11} - k_4y_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)} + \dots \\ \mathbf{T}(\zeta) = \mathbf{T}_0 + \dfrac{z_{11}\zeta^{\varrho}}{\Gamma(\varrho+1)} + [\xi^{\varrho}y_{11} - k_5z_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)} + \dots \\ \mathbf{R}(\zeta) = \mathbf{R}_0 + \dfrac{a_{11}\zeta^{\varrho}}{\Gamma(\varrho+1)} + [\gamma^{\varrho}w_{11} + \mu^{\varrho}z_{11} - k_6a_{11}]\dfrac{\zeta^{2\varrho}}{\Gamma(2\varrho+1)} + \dots \end{cases}$$

The unknown values of the above equations are,

$$\begin{cases} u_{11} = (1-\pi^{\varrho})\Lambda^{\varrho} + \rho^{\varrho}N_2 + (1-r_1)\lambda^{\varrho}N_3 + \sigma^{\varrho}N_7 - \alpha^{\varrho}N_1(N_5 + \beta^{\varrho}N_6) - \eta^{\varrho}N_1, \\ v_{11} = \pi^{\varrho}\Lambda^{\varrho} - k_1N_2, \qquad w_{11} = \alpha^{\varrho}N_1(N_5 + \beta^{\varrho}N_6) - k_2N_3, \quad x_{11} = r_1\lambda^{\varrho}N_3 - k_3N_4, \\ y_{11} = \omega^{\varrho}N_3 + \theta^{\varrho}N_4 - k_4N_5, \quad z_{11} = \xi^{\varrho}N_5 - k_5N_6, \qquad a_{11} = \gamma^{\varrho}N_3 + \mu^{\varrho}N_6 - k_6N_7. \end{cases}$$

8.2 The construction of solution for Model (2) by FDTM

In this section, we apply FDTM to solve the proposed Model (2) with time ζ. Transforming the proposed model by Theorems 3.1, 3.2, 3.3 and 3.4, we have,

$$\begin{cases} \mathbf{S}(\tau+\varrho m) = \dfrac{\Gamma(1+\frac{\tau}{m})}{\Gamma(\varrho+1+\frac{\tau}{m})}[(1-\pi^{\varrho})\Lambda^{\varrho}+\rho^{\varrho}\mathbf{M}(\tau)+(1-r_1)\lambda^{\varrho}\mathbf{E}(\tau)+\sigma^{\varrho}\mathbf{R}(\tau) \\ \qquad\qquad -\alpha^{\varrho}\sum_{n=0}^{\tau}\mathbf{S}(\tau)\mathbf{I}(\tau-n)-\alpha^{\varrho}\beta^{\varrho}\sum_{n=0}^{\tau}\mathbf{S}(\tau)\mathbf{T}(\tau-n)-\eta^{\varrho}\mathbf{S}(\tau)], \\ \mathbf{M}(\tau+\varrho m) = \dfrac{\Gamma(1+\frac{\tau}{m})}{\Gamma(\varrho+1+\frac{\tau}{m})}[\pi^{\varrho}\Lambda^{\varrho}-k_1\mathbf{M}(\tau)], \\ \mathbf{E}(\tau+\varrho m) = \dfrac{\Gamma(1+\frac{\tau}{m})}{\Gamma(\varrho+1+\frac{\tau}{m})}[\alpha^{\varrho}\sum_{n=0}^{\tau}\mathbf{S}(\tau)\mathbf{I}(\tau-n)+\alpha^{\varrho}\beta^{\varrho}\sum_{n=0}^{\tau}\mathbf{S}(\tau)\mathbf{T}(\tau-n)-k_2\mathbf{E}(\tau)], \\ \mathbf{A}(\tau+\varrho m) = \dfrac{\Gamma(1+\frac{\tau}{m})}{\Gamma(\varrho+1+\frac{\tau}{m})}[r_1\lambda^{\varrho}\mathbf{E}(\tau)-k_3\mathbf{A}(\tau)], \\ \mathbf{I}(\tau+\varrho m) = \dfrac{\Gamma(1+\frac{\tau}{m})}{\Gamma(\varrho+1+\frac{\tau}{m})}[\omega^{\varrho}\mathbf{E}(\tau)+\theta^{\varrho}\mathbf{A}(\tau)-k_4\mathbf{I}(\tau)], \\ \mathbf{T}(\tau+\varrho m) = \dfrac{\Gamma(1+\frac{\tau}{m})}{\Gamma(\varrho+1+\frac{\tau}{m})}[\xi^{\varrho}\mathbf{I}(\tau)-k_5\mathbf{T}(\tau)], \\ \mathbf{R}(\tau+\varrho m) = \dfrac{\Gamma(1+\frac{\tau}{m})}{\Gamma(\varrho+1+\frac{\tau}{m})}[\gamma^{\varrho}\mathbf{E}(\tau)+\mu^{\varrho}\mathbf{T}(\tau)-k_6\mathbf{R}(\tau)], \end{cases} \tag{24}$$

where τ is the fraction of order ϱ. The initial conditions are $\mathbf{S}(0)=N_1$, $\mathbf{M}(0)=N_2$, $\mathbf{E}(0)=N_3$, $\mathbf{A}(0)=N_4$, $\mathbf{I}(0)=N_5$, $\mathbf{T}(0)=N_6$, and $\mathbf{R}(0)=N_7$. The fractional differential transform of $\mathbf{S}(\tau+\varrho m)$, $\mathbf{M}(\tau+\varrho m)$, $\mathbf{E}(\tau+\varrho m)$, $\mathbf{A}(\tau+\varrho m)$, $\mathbf{I}(\tau+\varrho m)$, $\mathbf{T}(\tau+\varrho m)$, and $\mathbf{R}(\tau+\varrho m)$ is defined as: When ζ_0 is taken as zero, the given function $\mathbf{S}(\zeta)$, $\mathbf{M}(\zeta)$, $\mathbf{E}(\zeta)$, $\mathbf{A}(\zeta)$, $\mathbf{I}(\zeta)$, $\mathbf{T}(\zeta)$ and $\mathbf{R}(\zeta)$ is declared by the following and the above equation can be written in the form,

$$\begin{cases} \mathbf{S}(\zeta)=\sum_{\tau=0}^{\infty}\mathbf{S}(\tau)\zeta^{\tau\varrho}, \quad \mathbf{M}(\zeta)=\sum_{\tau=0}^{\infty}\mathbf{M}(\tau)\zeta^{\tau\varrho}, \quad \mathbf{E}(\zeta)=\sum_{\tau=0}^{\infty}\mathbf{E}(\tau)\zeta^{\tau\varrho}, \\ \mathbf{A}(\zeta)=\sum_{\tau=0}^{\infty}\mathbf{A}(\tau)\zeta^{\tau\varrho}, \quad \mathbf{I}(\zeta)=\sum_{\tau=0}^{\infty}\mathbf{I}(\tau)\zeta^{\tau\varrho}, \quad \mathbf{T}(\zeta)=\sum_{\tau=0}^{\infty}\mathbf{T}(\tau)\zeta^{\tau\varrho}, \\ \mathbf{R}(\zeta)=\sum_{\tau=0}^{\infty}\mathbf{R}(\tau)\zeta^{\tau\varrho}. \end{cases} \tag{25}$$

By solving (24) and (25) for $\mathbf{S}(\tau+\varrho m)$, $\mathbf{M}(\tau+\varrho m)$, $\mathbf{E}(\tau+\varrho m)$, $\mathbf{A}(\tau+\varrho m)$, $\mathbf{I}(\tau+\varrho m)$, $\mathbf{T}(\tau+\varrho m)$, and $\mathbf{R}(\tau+\varrho m)$ up to order 4, we get the function of $\mathbf{S}(\zeta)$, $\mathbf{M}(\zeta)$, $\mathbf{E}(\zeta)$, $\mathbf{A}(\zeta)$, $\mathbf{I}(\zeta)$, $\mathbf{T}(\zeta)$, and $\mathbf{R}(\zeta)$ respectively,

$$\begin{cases} \mathbf{S}(\zeta)=\sum_{\tau=0}^{3}\mathbf{S}(\tau)\zeta^{\tau\varrho}, \quad \mathbf{M}(\zeta)=\sum_{\tau=0}^{3}\mathbf{M}(\tau)\zeta^{\tau\varrho}, \quad \mathbf{E}(\zeta)=\sum_{\tau=0}^{3}\mathbf{E}(\tau)\zeta^{\tau\varrho}, \\ \mathbf{A}(\zeta)=\sum_{\tau=0}^{3}\mathbf{A}(\tau)\zeta^{\tau\varrho}, \quad \mathbf{I}(\zeta)=\sum_{\tau=0}^{3}\mathbf{I}(\tau)\zeta^{\tau\varrho}, \quad \mathbf{T}(\zeta)=\sum_{\tau=0}^{3}\mathbf{T}(\tau)\zeta^{\tau\varrho}, \\ \mathbf{R}(\zeta)=\sum_{\tau=0}^{3}\mathbf{R}(\tau)\zeta^{\tau\varrho}. \end{cases} \tag{26}$$

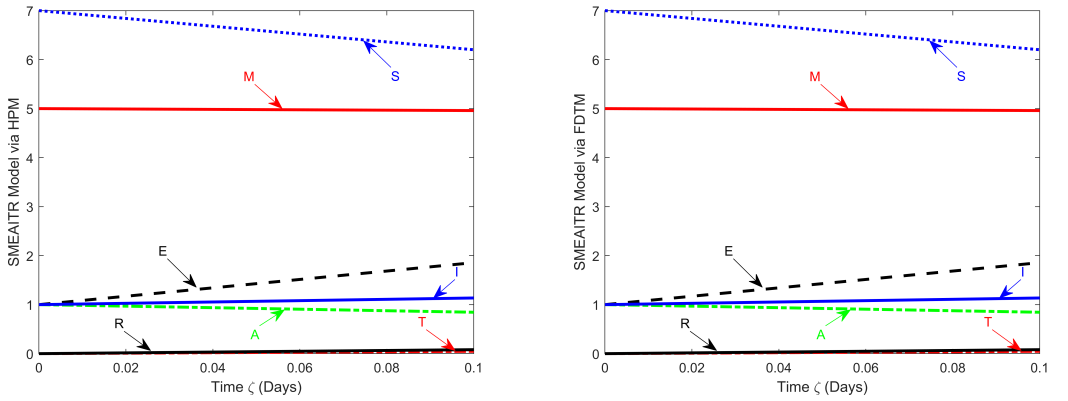
From (24), (25) and (26), we have,

$$\begin{aligned}
\mathbf{S}(\zeta) =&\mathbf{S}(0)+\frac{1}{\Gamma(\varrho+1)}[(1-\pi^{\varrho})\Lambda^{\varrho}+\rho^{\varrho}\mathbf{M}(0)+(1-r_1)\Lambda^{\varrho}\mathbf{E}(0)+\sigma^{\varrho}\mathbf{R}(0)-\alpha^{\varrho}\mathbf{S}(0)\mathbf{I}(0)\\
&-\alpha^{\varrho}\beta^{\varrho}\mathbf{S}(0)\mathbf{T}(0)-\eta^{\varrho}\mathbf{S}(0)]\zeta^{\varrho}+\frac{\Gamma(1+\frac{1}{m})}{\Gamma(\varrho+1+\frac{1}{m})}[(1-\pi^{\varrho})\Lambda^{\varrho}+\rho^{\varrho}\mathbf{M}(1)+(1-r_1)\Lambda^{\varrho}\mathbf{E}(1)\\
&+\sigma^{\varrho}\mathbf{R}(1)-\alpha^{\varrho}\sum_{n=0}^{1}\mathbf{S}(1)\mathbf{I}(1-n)-\alpha^{\varrho}\beta^{\varrho}\sum_{n=0}^{1}\mathbf{S}(1)\mathbf{T}(1-n)-\eta^{\varrho}\mathbf{S}(1)]\zeta^{2\varrho}\\
&+\frac{\Gamma(1+\frac{2}{m})}{\Gamma(\varrho+1+\frac{2}{m})}[(1-\pi^{\varrho})\Lambda^{\varrho}+\rho^{\varrho}\mathbf{M}(2)+(1-r_1)\Lambda^{\varrho}\mathbf{E}(2)+\sigma^{\varrho}\mathbf{R}(2)\\
&-\alpha^{\varrho}\sum_{n=0}^{2}\mathbf{S}(2)\mathbf{I}(2-n)-\alpha^{\varrho}\beta^{\varrho}\sum_{n=0}^{2}\mathbf{S}(2)\mathbf{T}(2-n)-\eta^{\varrho}\mathbf{S}(2)]\zeta^{3\varrho}\\
&+\frac{\Gamma(1+\frac{3}{m})}{\Gamma(\varrho+1+\frac{3}{m})}[(1-\pi^{\varrho})\Lambda^{\varrho}+\rho^{\varrho}\mathbf{M}(3)+(1-r_1)\Lambda^{\varrho}\mathbf{E}(3)+\sigma^{\varrho}\mathbf{R}(3)\\
&-\alpha^{\varrho}\sum_{n=0}^{3}\mathbf{S}(3)\mathbf{I}(3-n)-\alpha^{\varrho}\beta^{\varrho}\sum_{n=0}^{3}\mathbf{S}(3)\mathbf{T}(3-n)-\eta^{\varrho}\mathbf{S}(3)]\zeta^{4\varrho}.\\
\mathbf{M}(\zeta) =&\mathbf{M}(0)+\frac{1}{\Gamma(\varrho+1)}[\pi^{\varrho}\Lambda^{\varrho}-k_1\mathbf{M}(0)]\zeta^{\varrho}+\frac{\Gamma(1+\frac{1}{m})}{\Gamma(\varrho+1+\frac{1}{m})}[\pi^{\varrho}\Lambda^{\varrho}-k_1\mathbf{M}(1)]\zeta^{2\varrho}\\
&+\frac{\Gamma(1+\frac{2}{m})}{\Gamma(\varrho+1+\frac{2}{m})}[\pi^{\varrho}\Lambda^{\varrho}-k_1\mathbf{M}(2)]\zeta^{3\varrho}+\frac{\Gamma(1+\frac{3}{m})}{\Gamma(\varrho+1+\frac{3}{m})}[\pi^{\varrho}\Lambda^{\varrho}-k_1\mathbf{M}(3)]\zeta^{4\varrho}.\\
\mathbf{E}(\zeta) =&\mathbf{E}(0)+\frac{1}{\Gamma(\varrho+1)}[\alpha^{\varrho}\mathbf{S}(0)\mathbf{I}(0)+\alpha^{\varrho}\beta^{\varrho}\mathbf{S}(0)\mathbf{T}(0)-k_2\mathbf{E}(0)]\zeta^{\varrho}\\
&+\frac{\Gamma(1+\frac{1}{m})}{\Gamma(\varrho+1+\frac{1}{m})}[\alpha^{\varrho}\sum_{n=0}^{1}\mathbf{S}(1)\mathbf{I}(1-n)+\alpha^{\varrho}\beta^{\varrho}\sum_{n=0}^{1}\mathbf{S}(1)\mathbf{T}(1-n)-k_2\mathbf{E}(1)]\zeta^{2\varrho}\\
&+\frac{\Gamma(1+\frac{2}{m})}{\Gamma(\varrho+1+\frac{2}{m})}[\alpha^{\varrho}\sum_{n=0}^{2}\mathbf{S}(2)\mathbf{I}(2-n)+\alpha^{\varrho}\beta^{\varrho}\sum_{n=0}^{2}\mathbf{S}(2)\mathbf{T}(2-n)-k_2\mathbf{E}(2)]\zeta^{3\varrho}\\
&+\frac{\Gamma(1+\frac{3}{m})}{\Gamma(\varrho+1+\frac{3}{m})}[\alpha^{\varrho}\sum_{n=0}^{3}\mathbf{S}(3)\mathbf{I}(3-n)+\alpha^{\varrho}\beta^{\varrho}\sum_{n=0}^{3}\mathbf{S}(3)\mathbf{T}(3-n)-k_2\mathbf{E}(3)]\zeta^{4\varrho}.\\
\mathbf{A}(\zeta) =&\mathbf{A}(0)+\frac{1}{\Gamma(\varrho+1)}[r_1\lambda^{\varrho}\mathbf{E}(0)-k_3\mathbf{A}(0)]\zeta^{\varrho}+\frac{\Gamma(1+\frac{1}{m})}{\Gamma(\varrho+1+\frac{1}{m})}[r_1\lambda^{\varrho}\mathbf{E}(1)-k_3\mathbf{A}(1)]\zeta^{2\varrho}\\
&+\frac{\Gamma(1+\frac{2}{m})}{\Gamma(\varrho+1+\frac{2}{m})}[r_1\lambda^{\varrho}\mathbf{E}(2)-k_3\mathbf{A}(2)]\zeta^{3\varrho}+\frac{\Gamma(1+\frac{3}{m})}{\Gamma(\varrho+1+\frac{3}{m})}[r_1\lambda^{\varrho}\mathbf{E}(4)-k_3\mathbf{A}(4)]\zeta^{4\varrho}.\\
\mathbf{I}(\zeta) =&\mathbf{I}(0)+\frac{1}{\Gamma(\varrho+1)}[\omega^{\varrho}\mathbf{E}(0)+\theta^{\varrho}\mathbf{A}(0)-k_4\mathbf{I}(0)]\zeta^{\varrho}+\frac{\Gamma(1+\frac{1}{m})}{\Gamma(\varrho+1+\frac{1}{m})}[\omega^{\varrho}\mathbf{E}(1)+\theta^{\varrho}\mathbf{A}(1)\\
&-k_4\mathbf{I}(1)]\zeta^{2\varrho}+\frac{\Gamma(1+\frac{2}{m})}{\Gamma(\varrho+1+\frac{2}{m})}[\omega^{\varrho}\mathbf{E}(2)+\theta^{\varrho}\mathbf{A}(2)-k_4\mathbf{I}(2)]\zeta^{3\varrho}+\frac{\Gamma(1+\frac{3}{m})}{\Gamma(\varrho+1+\frac{3}{m})}[\omega^{\varrho}\mathbf{E}(3)\\
&+\theta^{\varrho}\mathbf{A}(3)-k_4\mathbf{I}(3)]\zeta^{4\varrho}.
\end{aligned}$$

$$\mathbf{T}(\zeta) = \mathbf{T}(0) + \frac{1}{\Gamma(\varrho+1)}[\xi^{\varrho}\mathbf{I}(0) - k_5\mathbf{T}(0)]\zeta^{\varrho} + \frac{\Gamma(1+\frac{1}{m})}{\Gamma(\varrho+1+\frac{1}{m})}[\xi^{\varrho}\mathbf{I}(1) - k_5\mathbf{T}(1)]\zeta^{2\varrho}$$
$$+ \frac{\Gamma(1+\frac{2}{m})}{\Gamma(\varrho+1+\frac{2}{m})}[\xi^{\varrho}\mathbf{I}(2) - k_5\mathbf{T}(2)]\zeta^{3\varrho} + \frac{\Gamma(1+\frac{3}{m})}{\Gamma(\varrho+1+\frac{3}{m})}[\xi^{\varrho}\mathbf{I}(3) - k_5\mathbf{T}(3)]\zeta^{4\varrho}.$$
$$\mathbf{R}(\zeta) = \mathbf{R}(0) + \frac{1}{\Gamma(\varrho+1)}[\gamma^{\varrho}\mathbf{E}(0) + \mu^{\varrho}\mathbf{T}(0) - k_6\mathbf{R}(0)]\zeta^{\varrho} + \frac{\Gamma(1+\frac{1}{m})}{\Gamma(\varrho+1+\frac{1}{m})}[\gamma^{\varrho}\mathbf{E}(1) + \mu^{\varrho}\mathbf{T}(1)$$
$$- k_6\mathbf{R}(1)]\zeta^{2\varrho} + \frac{\Gamma(1+\frac{2}{m})}{\Gamma(\varrho+1+\frac{2}{m})}[\gamma^{\varrho}\mathbf{E}(2) + \mu^{\varrho}\mathbf{T}(2) - k_6\mathbf{R}(2)]\zeta^{3\varrho} + \frac{\Gamma(1+\frac{3}{m})}{\Gamma(\varrho+1+\frac{3}{m})}[\gamma^{\varrho}\mathbf{E}(3)$$
$$+ \mu^{\varrho}\mathbf{T}(3) - k_6\mathbf{R}(3)]\zeta^{4\varrho}.$$

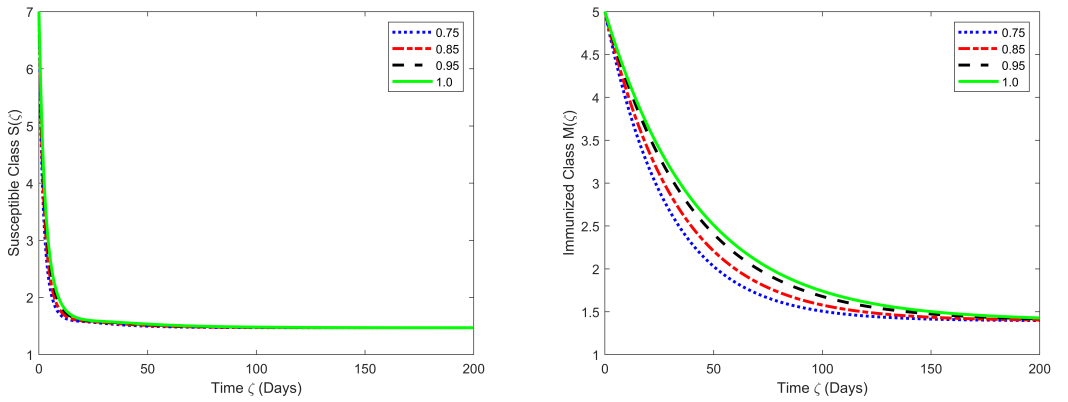
8.3 Graphical representation

In this section, we examine how well our established iterative approach and the chosen fractional derivative align with the graphs behaviors. We utilize the data provided in Table 1 for this analysis. We simulate all compartments for various fractional orders and an integer order.

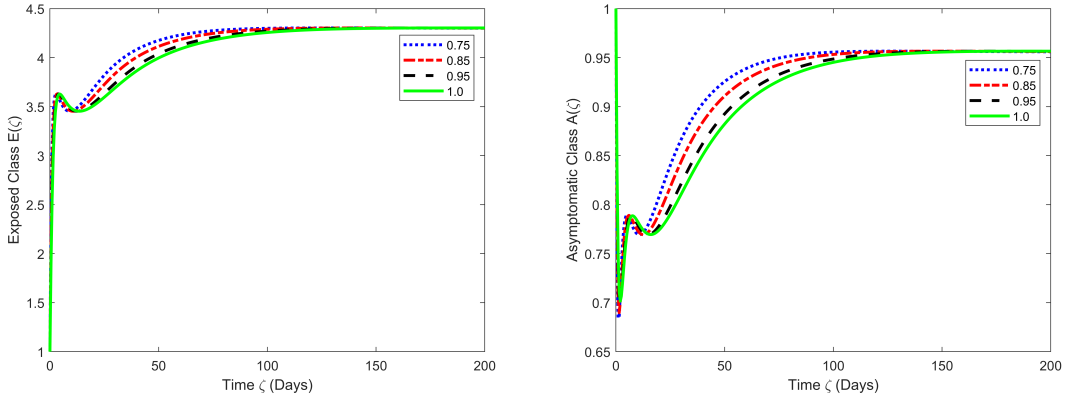


(a) (b)

Figure 3: Graphical comparison of approximate solution of the HPM and FDTM for the Model (2) at $\varrho = 1$.

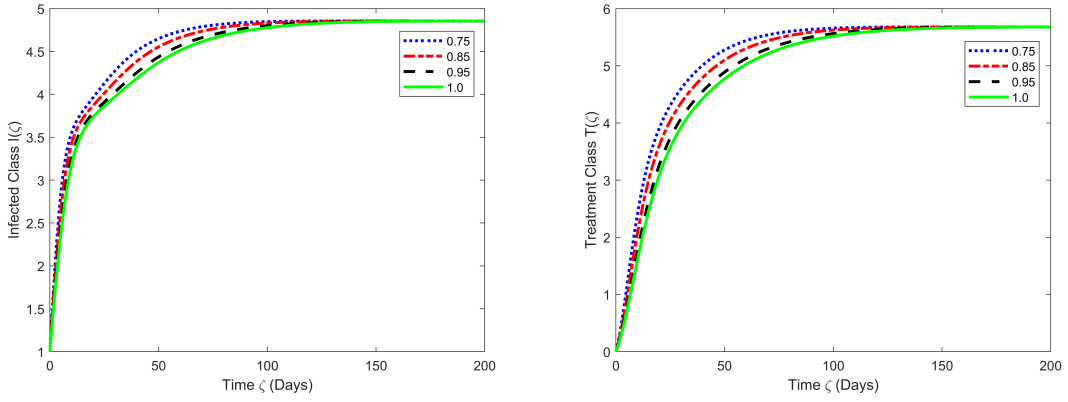


(a) (b)



(c) (d)

Figure 4: Graphical representation of the approximate results of susceptible, immunized, exposed, and asymptomatic classes for orders $\varrho = 1, 0.95, 0.85, 0.75$.



(a) (b)

Figure 5: Graphical representation of the approximate results of infected, and treatment classes for orders $\varrho = 1, 0.95, 0.85, 0.75$.

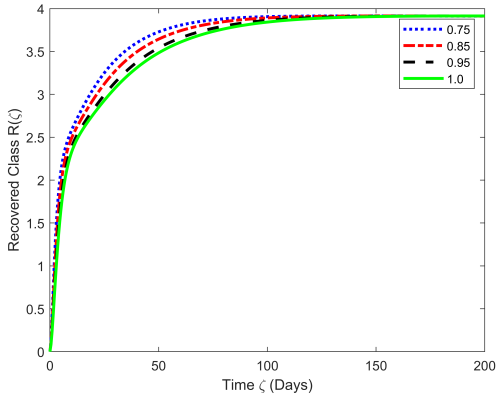


Figure 6: Graphical representation of the approximate results of recovered class for orders $\varrho = 1, 0.95, 0.85, 0.75$.

First, we present comparison graphs in Figure 3(a)-3(b) to evaluate different compartments of the model for $\varrho = 1$. The compartments results obtain through the FDTM and HPM techniques. The simulations exhibit similarity, indicating that both schemes yield consistent results. The HPM stands out as a superior choice for solving fractional order mathematical models of infectious diseases due to its ability to handle strong non–linearity, simplicity and flexibility, rapid convergence, broad applicability, and effectiveness. These advantages make HPM a powerful tool for accurately modelling and predicting the dynamics of infectious diseases [27]. Due to these advantages, we choose HPM to find the solution of the model (2) at different fractional orders.

Figure 4(a) shows the susceptible class is decreasing with the time increases. This is due to self–immune against new epidemic infections. The immunized class decreases with different fractional orders at varying rates, as depicted in Figure 4(b). The exposed class initially decreases and increases due to change in the time (days) as presented in Figure 4(c). This is due to an increase in asymptomatic individuals. After 150 days, exposed class shows stability. Figures 4(d) and 5(a) illustrate the decline and growth of the asymptomatic and infected classes, indicating a rise in sickness or innate affliction within the community. The proportion of cured individuals increases with the adoption of appropriate treatment and recovered classes as shown in Figure 5(b) and Figure 6 respectively.

The rate at which the population grows or declines changes more rapidly at lower fractional orders, but this pattern reverses as the fractional order increases, with higher fractional orders leading to faster population growth or decline in a specific class. The figures demonstrate the stability and convergence of the model classes. From the graphical observation, the health improvement can be expected in this population over time. The convergence of the curves for fractional orders is observed to be faster as compared to the integer order 1, attributes to the robustness of our suggested model. Moreover, we observed from the graphical results, that the recovered class grows due to self–immunity and treatment, while the susceptible class declines rapidly. It means strong immune system and appropriate treatment of infected individuals can cause to decrease the effects of disease.

9 Conclusions

This study highlights the transmission dynamics of TB using a mathematical model with the Caputo fractional derivative, incorporating immune and asymptomatic individuals. We proved the existence and uniqueness of solutions using fixed point theory and analyzed Hyers–Ulam stability. Sensitivity analysis showed that reducing the parameter θ lowered the number of infected cases. Numerical simulations were carried out by using the HPM and FDTM. Unlike traditional methods that require discretization or simplification, HPM provides quick and accurate solutions without approximations, while FDTM effectively avoids rounding errors. Both methods are ideal for solving complex nonlinear problems involving advanced mathematical concepts. Graphical representations demonstrated the decline in the number of asymptomatic and susceptible classes due to self–immune response and appropriate treatment, suggesting that a healthy diet and supportive care for those who are infected can help to reduce the disease effects.

In future work, we will use this research to analyze the pandemics in other states. In order to give decision–makers more effective methods for halting the spread of TB, our future research may potentially take optimal control theory into account.

Acknowledgement The authors are also thankful to the referees for their useful comments and suggestions.

Conflicts of Interest The authors declare no conflict of interest.

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